THE GELFAND-GRAEV REPRESENTATION OF $\mathrm{GSp}(4, \mathbb{F}_q)$

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Abstract. In this article, we demonstrate a technique for calculating the irreducible representations of the endomorphism algebra of the Gelfand-Graev representations of the general symplectic group $\mathrm{GSp}(4, \mathbb{F}_q)$, where q is an odd prime power.

Contents

1. Background

1.1. **Introduction and notation.** Let $\mathbb{F}_q$ denote a finite field with $q = p^n$ elements, where p is an odd prime. Let G be the genus two symplectic or general symplectic group over $\mathbb{F}_q$. Given any non-trivial linear character ψ of a maximal unipotent subgroup of G, the Gelfand-Graev representation is the induced representation of ψ to G [3, Chapter 8]. In this article we examine the representations of the Hecke algebra, $\mathcal{H}$, of this Gelfand-Graev representation. Our approach is to use a theorem of Curtis [4] which tells us that any irreducible character χ of $\mathcal{H}$ can be factored as $\chi = \bar{\theta} \circ f_T$, where $\bar{\theta}$ is an extension to $\mathbb{C}T$ of a character θ of a maximal torus T of G and $f_T : \mathcal{H} \to \mathbb{C}T$ is a unique homomorphism which is independent of θ.

The layout of this article is as follows. In this section, we describe a basis of this Hecke algebra $\mathcal{H}$ and provide some background and data needed to begin determining the characters of $\mathcal{H}$. In Section 2, we determine the semisimple and unipotent parts of un for certain u in a unipotent radical inside a Borel subgroup of G and certain coset representatives n of $N_G(T)/T$ for a maximal split torus T inside this Borel subgroup. Section 3 is then an illustration of how the data from Sections 1 and 2 can be pulled together to provide the irreducible characters of $\mathcal{H}$.

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Definition 1.1. The *general symplectic group of genus* 2 *over* $\mathbb{F}_q$, denoted $\mathrm{GSp}(4,\mathbb{F}_q)$, is defined as

$$\mathrm{GSp}(4,\mathbb{F}_q)=\{g\in \mathrm{GL}(4,\mathbb{F}_q): {}^tgJg=d(g)J\},\text{ where } J=\begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix}$$

for some $d(g)\in\mathbb{F}_q^\times$. The element $d(g)$ is called the *multiplier* of g. The set of all $g\in\mathrm{GSp}(4,\mathbb{F}_q)$ such that $d(g)=1$ is a subgroup called the *symplectic group of genus* 2 *over* $\mathbb{F}_q$ and is denoted by $\mathrm{Sp}(4,\mathbb{F}_q)$.

The order of $\mathrm{Sp}(4,\mathbb{F}_q)$ is $q^4(q^4-1)(q^2-1)$, see [8] or [9]. The order of $\mathrm{GSp}(4,\mathbb{F}_q)$ is then $q^4(q^4-1)(q^2-1)(q-1)$ since any $g\in\mathrm{GSp}(4,\mathbb{F}_q)$ can be written uniquely as

$$g=\begin{pmatrix}1&&&\\&1&&\\&&d(g)&\\&&&d(g)\end{pmatrix}\cdot g',$$

for some $g'\in\mathrm{Sp}(4,\mathbb{F}_q)$.

For the remainder of this article, G will denote either $\mathrm{GSp}(4,\mathbb{F}_q)$ or $\mathrm{Sp}(4,\mathbb{F}_q)$. A standard choice for a Borel subgroup B of G is the set of all of the upper triangular matrices,

$$B=\left\{\begin{pmatrix}*&*&*&*\\&*&*&*\\&&*&*\\&&&*\end{pmatrix}\in G\right\}.$$

Every element $x\in B$ can be written uniquely in the form

$$x=\begin{pmatrix}a&&&\\&b&&\\&&cb^{-1}&\\&&&ca^{-1}\end{pmatrix}\cdot\begin{pmatrix}1&&&\\&1&h&\\&&1&\\&&&1\end{pmatrix}\cdot\begin{pmatrix}1&d&e&f\\&1&&e\\&&1&-d\\&&&1\end{pmatrix},$$

with $a,b,c\in\mathbb{F}_q^\times$ and $d,e,f,h\in\mathbb{F}_q$. The subgroup of diagonal matrices

$$t(a,b,c)=\begin{pmatrix}a&&&\\&b&&\\&&cb^{-1}&\\&&&ca^{-1}\end{pmatrix}$$

in G will be denoted by T_o. This subgroup T_o is the *split torus* of G. The unipotent radical U of B is the set of elements of B of the form

$$u(h,f,d,e)=\begin{pmatrix}1&&&\\&1&h&\\&&1&\\&&&1\end{pmatrix}\cdot\begin{pmatrix}1&d&e&f\\&1&&e\\&&1&-d\\&&&1\end{pmatrix}.$$

Let $N(T_o)$ be the normalizer in G of T_o. The Weyl group $W=N(T_o)/T_o$ of G is a group of order 8 and is generated by the images in W of the following two elements of $N(T_o)$:

$$s_1=\begin{pmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{pmatrix},\qquad s_2=\begin{pmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{pmatrix}.$$

1.2. **Hecke algebras of Gelfand-Graev representations and Curtis' Theorem.** Let ψ_1 and ψ_2 be non-trivial linear characters of the additive group $\mathbb{F}_q$ and let ψ be the character of U defined by

$$\psi\begin{pmatrix}1&d&*&*\\&1&h&*\\&&1&-d\\&&&1\end{pmatrix}=\psi_1(h)\psi_2(d).$$

The representation of G induced from ψ is its Gelfand-Graev representation. The values of $\mathrm{Ind}_U^G\psi$ are given in [1] and [2]. Moreover, the irreducible characters of $\mathrm{Sp}(4,\mathbb{F}_q)$ can be found in [8] and the irreducible characters of $\mathrm{GSp}(4,\mathbb{F}_q)$ can be found in [1], [2], and [7]. Define

$$e_\psi = \frac{1}{|U|}\sum_{u\in U}\psi(u^{-1})u.$$

Note e_ψ is an element of the group algebra $\mathbb{C}G$ and e_ψ is called the *primitive central idempotent affording the representation* ψ. The Hecke algebra $\mathcal{H}_\psi$ of the induced character $\mathrm{Ind}_U^G(\psi)$ is defined by

$$\mathcal{H}_\psi = e_\psi \mathbb{C}G e_\psi.$$

For any two non-trivial linear characters χ_1,χ_2 of U, the induced characters $\mathrm{Ind}_U^G\chi_1, \mathrm{Ind}_U^G\chi_2$ are equivalent and so we will drop the subscripts and write $e = e_\psi$ and $\mathcal{H} = \mathcal{H}_\psi$.

Let N be a cross section of the double cosets $U\backslash G/U$. For $n\in N$, let

$$\mathrm{ind}\, n = |UnU|/|U| = |U : U\cap {}^nU|$$

and let

$$c_n = (\mathrm{ind}\, n)e_\psi n e_\psi.$$

The standard basis of $\mathcal{H}_\psi$ (see [4]) is

$$\mathcal{B} = \{c_n \mid n\in N \text{ and } \psi(u) = {}^n\psi(u) \text{ for all } u\in U\cap {}^nU\}.$$

By the Bruhat decomposition, the elements $n\in N$ are of the form $n = t\cdot w$, where $t = t(a,b,c)\in T$ and $wT\in W$, i.e., $w\in N(T)$ is a coset representative. In what follows, we will refer to these coset representatives w, with a slight abuse of terminology, as Weyl group elements. The intersections $U\cap {}^nU$ are given in Table 1 for all such elements $n\in N$.

It is now straightforward to determine whether or not $\mathrm{ind}(n)ene$ is an element of the standard basis of $\mathcal{H}$ by checking if $\psi = {}^n\psi$ on $U\cap {}^nU$. So we have the following two theorems.

Theorem 1.2. *Let $G = \mathrm{GSp}(4,\mathbb{F}_q)$. Using the notation described above, the standard basis for the Hecke algebra $\mathcal{H} = e\mathbb{C}Ge$ of G is the set of elements $\mathrm{ind}(n)ene$ such that n is one of the four below types of elements.*

- *Type 0: $n = t(a,a,a^2)$, where $a\in\mathbb{F}_q^\times$ and in this case $\mathrm{ind}\, n = 1$.*
- *Type 1: $n = t(a,b,c)\cdot(s_1s_2)^2$, where $a,b,c\in\mathbb{F}_q^\times$ and in this case $\mathrm{ind}\, n = q^4$.*
- *Type 2: $n = t(a,-a,c)\cdot s_2s_1s_2$, where $a,c\in\mathbb{F}_q^\times$ and in this case $\mathrm{ind}\, n = q^3$.*

- *Type 3:* $n = t(a,b,b^2)\cdot s_1s_2s_1$, *where* $a,b\in\mathbb{F}_q^\times$ *and in this case* $\operatorname{ind} n = q^3$.

Proof. First assume $n = ts_1$. That is $n = t(a,b,c)s_1$ for some $a,b,c\in\mathbb{F}_q^\times$. Let $u\in {}^nU\cap U$. Then $u = u(h,f,0,e)$ for some $h,f,e\in\mathbb{F}_q$ by Table 1 and thus $\psi(u) = \psi_2(h)$. However $nun^{-1} = u(\frac{b^2f}{c},\frac{a^2h}{c},0,\frac{aeb}{c})$ so that ${}^n\psi(u) = \psi_2(\frac{b^2f}{c})$. Thus, since we need $\psi = {}^n\psi$ for all $u\in U\cap{}^nU$, we must have $h = \frac{b^2f}{c}$ for all $h,f\in\mathbb{F}_q$. This has no solution. Thus $\operatorname{ind}(n)ene$ is not a basis element of $\mathcal{H}$ when $n = ts_1$.

Now assume $n = ts_1s_2s_1$. That is $n = t(a,b,c)s_1s_2s_1$ for some $a,b,c\in\mathbb{F}_q^\times$. Let $u\in{}^nU\cap U$. Then $u = u(h,0,0,0)$ for some $h\in\mathbb{F}_q$ by Table 1 and thus $\psi(u) = \psi_2(h)$. However $nun^{-1} = u(\frac{b^2h}{c},0,0,0)$ so that ${}^n\psi(u) = \psi_2(\frac{b^2h}{c})$. Thus, since we need $\psi = {}^n\psi$ for all $u\in U\cap{}^nU$, we must have $h = \frac{b^2h}{c}$ for all $h\in\mathbb{F}_q$. Thus $b^2 = c$. So $\operatorname{ind}(n)ene$ is a basis element of $\mathcal{H}$ when $n = ts_1s_2s_1$ if and only if $t = t(a,b,b^2)$ for some $a,b\in\mathbb{F}_q^\times$. Also note in this cases the order of ${}^nU\cap U$ is q so that $\operatorname{ind}(n) = q^3$.

The proofs of the remaining cases are very similar and are omitted. □

Note for $G = \mathrm{Sp}(4,\mathbb{F}_q)$, the entries c of the diagonal matrices $t(a,b,c)$ are always equal to 1.

Theorem 1.3. *Let* $G = \mathrm{Sp}(4,\mathbb{F}_q)$. *The standard basis for the Hecke algebra* $\mathcal{H} = e\mathbb{C}Ge$ *of* G *is the set of elements* $\operatorname{ind}(n)ene$ *such that* n *is one of the four below types of elements.*

- *Type 0:* $n = t(\pm1,\pm1,1)$ *and in this case* $\operatorname{ind} n = 1$.
- *Type 1:* $n = t(a,b,1)\cdot(s_1s_2)^2$, *where* $a,b\in\mathbb{F}_q^\times$ *and in this case* $\operatorname{ind} n = q^4$.
- *Type 2:* $n = t(a,-a,1)\cdot s_2s_1s_2$, *where* $a\in\mathbb{F}_q^\times$ *and in this case* $\operatorname{ind} n = q^3$.
- *Type 3:* $n = t(a,\pm1,1)\cdot s_1s_2s_1$, *where* $a\in\mathbb{F}_q^\times$ *and in this case* $\operatorname{ind} n = q^3$.

Let $\tilde{T}$ be an F-stable torus of a connected reductive group $\tilde{G}$ defined over a finite field. Let θ be an irreducible character of the torus $T = \tilde{T}^F$ and let $R^G_{T,\theta}$ be the Deligne-Lusztig generalized character and let Q^G_T denote the Green function. For a unipotent element u, we have that $Q^G_T(u) = R^G_{T,\theta}(u)$ [3, p. 212]. The following theorem will be used frequently and will be referred to as Curtis' Theorem:

Theorem 1.4 ([4], 4.2)**.** *Let* $(\tilde{T},\theta)$, $\Gamma = \mathrm{Ind}_U^G(\psi)$, $\mathcal{H}$ *be given. Let* $\overline{\theta}$ *denote the extension of* θ *to* $\mathbb{C}T$. *For* $x\in G$, *let* x_s *and* x_u *denote the semisimple and unipotents parts of* x, *respectively. Then*

(1) *There exists a unique homomorphism* $f_T:\mathcal{H}\to\mathbb{C}T$ *which is independent of* θ *and has the property that each character* $f_{T,\theta}:\mathcal{H}\to\mathbb{C}$ *can be factored as* $f_{T,\theta} = \overline{\theta}\circ f_T$

(2)
$$f_T(c_n) = \sum_{t\in T} f_T(c_n)(t)t,$$

where c_n is an element in the standard basis of $\mathcal{H}$ and

$$f_T(c_n)(t) = \frac{\operatorname{ind} n}{< Q_T^G, \Gamma > |U|\,|\mathrm{C}_G(t)|} \sum_{\substack{g \in G,\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{\mathrm{C}_G(t)}((gung^{-1})_u),$$

where $\mathrm{C}_G(t)$ is the centralizer of t in G.

We will use Curtis' Theorem to demonstrate a technique for determining the irreducible representations of $\mathcal{H}$. Note that t and un have the same characteristic polynomial if and only if there exists $g \in G$ such that $(gung^{-1})_s = t$. So to compute the coefficients $f_T(c_n)(t)$ of t, we will first determine conditions on u and n such that un has the same characteristic polynomial as a given torus element t and then we will compute that values of the Green function on the unipotent parts of the element un.

1.3. **The maximal tori and Green functions of** $\mathrm{GSp}(4, \mathbb{F}_q)$**.** Given a connected reductive algebraic group $\tilde{G}$ and Frobenius endomorphism F on $\tilde{G}$, a maximal torus of $G = \tilde{G}^F$ is a subgroup of G of the form $\tilde{T}^F$, where $\tilde{T}$ is an F-stable maximal torus of the algebraic group $\tilde{G}$. Let w be a Weyl group element for G and let $T_w = \{A \in \tilde{T}_0 \mid wAw^{-1} = F(A)\}$, where $\tilde{T}_0$ denotes a maximal split F-stable torus of $\tilde{G}$. Then T_w is a maximal torus of G and every maximal torus of G is conjugate (in $\tilde{G}$) to some T_w. Furthermore, two maximal tori, T_{w_1} and T_{w_2}, of G are isomorphic if and only if w_1 and w_2 are in the same F-conjugacy class of W [3, Section 3.3]. So each maximal torus of G is isomorphic to

$$T_w = \{A \in \tilde{T}_0 \mid wAw^{-1} = F(A)\},$$

for some $w \in W$. In the case where the algebraic group is taken to be $\tilde{G} = \mathrm{GSp}(4, \bar{\mathbb{F}}_q)$ or $\mathrm{Sp}(4, \bar{\mathbb{F}}_q)$ and F is taken to be the standard Frobenius map (which raises each entry to the qth power), there are five F-conjugacy classes of the Weyl group W: $\{I\}$, $\{s_2, s_1s_2s_1\}$, $\{s_1, s_2s_1s_2\}$, $\{s_1s_2, s_2s_1\}$, and $\{(s_1s_2)^2\}$. Thus there are 5 maximal tori of G. The maximal torus corresponding to the conjugacy class of the identity of the Weyl group is the split torus T_o. The maximal tori of $G = \mathrm{GSp}(4, \mathbb{F}_q)$ are in the following list:

- $T_o = \{t(a, b, c) : a, b, c \in \mathbb{F}_q^\times\}$,
- $T_{s_1} = \left\{t(a, a^q, c) : a \in \mathbb{F}_{q^2}^\times, c \in \mathbb{F}_q^\times\right\}$,
- $T_{s_2} = \left\{t(a, b, b^{q+1}) : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_{q^2}^\times\right\}$,
- $T_{(s_1s_2)^2} = \left\{t(a, b, a^{q+1}) : a, b \in \mathbb{F}_{q^2}^\times, a^{q+1} = b^{q+1}\right\}$,
- $T_{s_2s_1} = \left\{t(a, a^q, a^{q^2+1}) : a^{(q^2+1)(q-1)} = 1\right\}$.

To get the maximal tori of $G = \mathrm{Sp}(4, \mathbb{F}_q)$ replace the entries c in the diagonal matrices $t(a, b, c)$ in this list by 1.

Let G be either $\mathrm{GSp}(4, \mathbb{F}_q)$ or $\mathrm{Sp}(4, \mathbb{F}_q)$ and let T be a maximal torus of G. Let θ be an irreducible character of T in general position. The Green function Q_T^G is defined on unipotent elements u of U by $Q_T^G(u) = R_{T,\theta}^G(u)$ where $R_{T,\theta}^G$ is the Deligne-Lusztig generalized character [3, Section 7.2]. In order to apply Curtis' Theorem, we will need to compute this Green function for each maximal torus T of G. We have that $Q_T^G(1) = \pm[G : T]_{p'}$ [3, Theorem 7.5.1]. In addition, if w is an element in the Weyl group of G and $T = T_w$, the sign of $Q_T^G(1)$ is equal to $(-1)^{\ell(w)}$, where $\ell(w)$ is the length of w [5, Remark 12.10]. Thus we have

$$\begin{aligned}
&Q_{T_o}^G(1) = (q^2+1)(q+1)^2,\\
&Q_{T_{s_1}}^G(1) = -(q^4-1),\\
&Q_{T_{s_2}}^G(1) = -(q^4-1),\\
&Q_{T_{(s_1s_2)^2}}^G(1) = (q^2+1)(q-1)^2,\\
&Q_{T_{s_2s_1}}^G(1) = (q^2-1)^2.
\end{aligned}$$

Furthermore, by Corollary 7.3.5 of [3], $R_{T,\theta}$ will be (up to sign) an irreducible character of G. The group G has five unipotent conjugacy classes. Using the notation for these conjugacy classes from [1] and [2], representatives for these five classes are

$$A_1 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1&0&0&0\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad A_{31} = \begin{pmatrix} 1&0&0&1\\ 0&1&-1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, A_{32} = \begin{pmatrix} 1&0&0&\gamma\\ 0&1&-1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1&1&-1&0\\ 0&1&-1&0\\ 0&0&1&-1\\ 0&0&0&1 \end{pmatrix},$$

where γ denotes a generator of the multiplicative group $\mathbb{F}_q^\times$. We remark that the matrices A_{31} and A_{32} are conjugate in $\mathrm{GL}(4, \mathbb{F}_q)$. An element $u = u(h, f, d, e) \in G$ that is conjugate in $\mathrm{GL}(4, \mathbb{F}_q)$ to both A_{31} and A_{32} will be in A_{31} if $h \in (\mathbb{F}_q^\times)^2$ otherwise $h \in A_{32}$. Here, $(\mathbb{F}_q^\times)^2$ denotes the squares in $\mathbb{F}_q^\times$.

On examination of the character tables of $\mathrm{GSp}(4, \mathbb{F}_q)$ and $\mathrm{Sp}(4, \mathbb{F}_q)$, we get the values of the Green function are as indicated in Table 2. We see that the Green functions for $\mathrm{GSp}(4, \mathbb{F}_q)$ and $\mathrm{Sp}(4, \mathbb{F}_q)$ are equal.

2. Semisimple and unipotent parts

We will now demonstrate a technique for computing the irreducible representations of $\mathcal{H}$ by computing the values of the coefficients $f_T(c_n)(t)$ for $T = T_o$ and n one of each of the four types of double coset representatives used in the computation of the standard basis of $\mathcal{H}$ (see Theorem 1.2). These coefficients are determined when n is equal to one of the following four elements:

$$n_0 = I, \qquad n_1 = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix}, \qquad n_2 = \begin{pmatrix} &&1&\\ &&&-1\\ 1&&&\\ &-1&& \end{pmatrix}, \qquad n_3 = \begin{pmatrix} &&&1\\ &1&&\\ &&1&\\ -1&&& \end{pmatrix}.$$

Note that en_0e is a basis element of type 0, en_1e is a basis element of type 1, en_2e is a basis element of type 2 and en_3e is a basis element of type 3 by Theorem 1.2. Since from now on we will be considering the case where $T = T_o$ we will drop the subscript and always refer to this split torus as just T.

2.1. **Semisimple parts of certain un and centralizers of $t \in T$.** Let $t = t(a, b, c) \in T$ and $u = u(h, f, d, e) \in U$. The following four lemmas describe the conditions on u and t so that t is the semisimple part of a conjugate of un. This is done by comparing the characteristic polynomials of un and t. For example, in Lemma 2.3 Case b, the characteristic polynomial of un_2 is

$$x^4 - hdx^3 + (hf + hde - e^2 - 2)x^2 - hdx + 1$$

and the characteristic polynomial of $t = (-a, a, a^2)$ is

$$x^4 - 2a^2x^2 + a^4.$$

Comparing the coefficients, we find the conditions

$$a^4 = 1, hd = 0, \text{and}\, e^2 - hf - 2a^2 + 2 = 0.$$

Lemma 2.1. *Let $t \in T$ and $u \in U$. Then $(gun_0g^{-1})_s = t$ for all $g \in G$ when $t = t(1, 1, 1)$. For $t \neq t(1, 1, 1)$ there does not exist any $u \in U$ or $g \in G$ such that $(gun_0g^{-1})_s = t$.*

Lemma 2.2. *Let $t \in T$ and $u \in U$. Then there exists $g \in G$ such that $(gun_1g^{-1})_s = t$ only under the following conditions.*

Case a: $t = t(a, a, a^2)$. *Then* $a^2 = 1$, $f = -(h + 4a)$, *and* $e^2 - hde + d^2 + (h + 2a)^2 = 0$.

Case b: $t = t(-a, a, a^2)$. *Then* $a^4 = 1, f = -h$, *and* $e^2 - hde + d^2 + h^2 - 2a^2 - 2 = 0$.

Case c: $t = t(a, a, c), c \neq a^2$. *Then* $c^2 = 1$. *If* $c = 1$, *then*

$$e^2 - hde + d^2 + \frac{(a^2 + ah + 1)^2}{a^2} = 0$$

and $f = -\frac{2a^2+ah+2}{a}$. *If* $c = -1$, *then* $a^2 = 1$, $e^2 - hde + d^2 + h^2 - 4 = 0$, *and* $f = -h$.

Case d: $t = t(a, b, b^2), b^2 \neq a^2$. *Then* $b^2 = 1$, $f = -\frac{a^2+(h+2b)a+1}{a}$, *and*

$$e^2 - hde + d^2 + \frac{(h + 2b)(a^2 + ah + 1)}{a} = 0.$$

Case e: $t = t(a, b, c), a \neq b, c \neq b^2$. *Then* $c^2 = 1$. *If* $c = 1$, *then* $f = -\frac{(a+b)(1+ab)+abh}{ab}$ *and*

$$e^2 - hde + d^2 + h^2 + \frac{(a + b)(1 + ab)h}{ab} + \frac{(a^2 + 1)(b^2 + 1)}{ab} = 0.$$

If $c = -1$, *then* $b = -a$ *or* $b = \frac{1}{a}$, $f = -h$, *and*

$$e^2 - hde + d^2 + h^2 - \frac{(a^2 + 1)^2}{a^2} = 0.$$

Lemma 2.3. *Let $t \in T$ and $u \in U$. Then there exists $g \in G$ such that $(gun_2g^{-1})_s = t$ only under the following conditions.*

Case a: $t = t(a,a,a^2)$. *Then* $a^2 = 1$, $d = \frac{4a}{h}$, *and* $(e-2a)^2 - hf + 4 = 0$.

Case b: $t = t(-a,a,a^2)$. *Then* $a^4 = 1, hd = 0$, *and* $e^2 - hf - 2a^2 + 2 = 0$.

Case c: $t = t(a,a,c), c \neq a^2$. *Then* $c^2 = 1$. *If* $c = 1$ *and* $h \neq 0$, *then* $\frac{(a^2-ae+1)^2}{a^2} - hf + 4 = 0$ *and* $d = \frac{2(a^2+1)}{ah}$. *If* $c = 1$ *and* $h = 0$, *then* $a^2 = -1$ *and* $e^2 + 4 = 0$. *If* $c = -1$, *then* $a^2 = 1, hd = 0$, *and* $e^2 - hf = 0$.

Case d: $t = t(a,b,b^2), b^2 \neq a^2$. *Then* $b^2 = 1$, $d = \frac{(a+b)^2}{ah}$, *and* $\frac{(e-2b)(a^2-ae+1)}{a} + hf - 4 = 0$.

Case e: $t = t(a,b,c), a \neq b, c \neq b^2$. *Then* $c^2 = 1$. *If* $c = 1$ *and* $h \neq 0$, *then* $d = \frac{(a+b)(1+ab)}{abh}$ *and*

$$\frac{(a^2-ae+1)(b^2-be+1)}{ab} - hf + 4 = 0.$$

If $c = 1$ *and* $h = 0$, *then* $e^2 - \frac{(a^2-1)^2}{a^2} = 0$ *and* $b = -a$ *or* $b = -\frac{1}{a}$. *If* $c = -1$ *and* $h \neq 0$, *then* $f = \frac{a^2e^2-(a^2-1)^2}{a^2h}$, $d = \frac{(a+b)(ab-1)}{abh}$, *and* $b = -a$ *or* $b = \frac{1}{a}$. *If* $c = -1$ *and* $h = 0$, *then* $e^2 - \frac{(a^2-1)^2}{a^2} = 0$ *and* $b = -a$ *or* $b = \frac{1}{a}$.

Lemma 2.4. *Let $t \in T$ and $u \in U$. Then there exists $g \in G$ such that $(gun_3g^{-1})_s = t$ only under the following conditions.*

Case a: $t = t(a,a,a^2)$. *Then* $a^2 = 1$. *If* $a = 1$, *then* $f = -2$ *and* $hd = 0$. *If* $a = -1$, *then* $h = -\frac{16}{d^2}$ *and* $f = 6$.

Case b: $t = t(-a,a,a^2)$. *Then* $a^4 = 1, f = 2$, *and* $hd^2 - 2a^2 + 2 = 0$.

Case c: $t = t(a,a,c), c \neq a^2$. *Then* $c^2 = 1$. *If* $c = 1$, *then* $f = -\frac{2(a^2-a+1)}{a}$ *and* $hd^2 + \frac{(a-1)^4}{a^2} = 0$. *If* $c = -1$, *then* $a^2 = 1$, $f = 2$, *and* $hd = 0$.

Case d: $t = t(a,b,b^2), b^2 \neq a^2$. *Then* $b^2 = 1$. *If* $b = 1$, *then* $f = -\frac{a^2+1}{a}$ *and* $hd = 0$. *If* $b = -1$, *then* $f = \frac{-a^2+4a-1}{a}$ *and* $hd^2 - \frac{4(a-1)^2}{a} = 0$.

Case e: $t = t(a,b,c), a \neq b, c \neq b^2$. *Then* $c^2 = 1$. *If* $c = 1$, *then* $f = \frac{2ab-(a+b)(1+ab)}{ab}$ *and* $hd^2 + \frac{(a-1)^2(b-1)^2}{ab} = 0$. *If* $c = -1$, *then* $f = 2$, $hd^2 - \frac{(a^2-1)^2}{a^2} = 0$, *and* $b = -a$ *or* $b = \frac{1}{a}$.

For each $t \in T$ such that there exists some $g \in G$ with $(gung^{-1})_s = t$ we need to know the subgroup $C_G(t)$, the order of $C_G(t)$ and the images of the Green function $Q_T^{C_G(t)}$. By the above results, the only such t are those given in Table 3. The orders are given under the assumption that $G = \mathrm{GSp}(4, \mathbb{F}_q)$.

In Table 3,

$$C_1 = \{\begin{pmatrix} g_{11} & & & g_{14} \\ & g_{22} & g_{23} & \\ & g_{32} & g_{33} & \\ g_{41} & & & g_{44} \end{pmatrix} \mid g_{11}g_{44} - g_{41}g_{14} = g_{22}g_{33} - g_{32}g_{23} \neq 0\}$$

$$C_2 = \{\begin{pmatrix} A & \\ & \lambda A' \end{pmatrix} \mid A \in \mathrm{GL}(2, \mathbb{F}_q), \lambda \in \mathbb{F}_q^\times\} \text{ where } A' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^tA^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

and

$$C_3 = \{\begin{pmatrix} \lambda & & \\ & A & \\ & & \lambda^{-1}\det(A) \end{pmatrix} \mid A \in \mathrm{GL}(2, \mathbb{F}_q), \lambda \in \mathbb{F}_q^\times\}.$$

When $t \in T$ is such that $C_G(t) = G$, the images of the Green function $Q_T^{C_G(t)} = Q_T^G$ were computed in subsection 1.3. When $t \in T$ is such that $C_G(t) = T$, then $Q_T^{C_G(t)} = Q_T^T$, whose images are all equal to 1. Note that C_i for $i = 1, 2, 3$ all have unipotent radical isomorphic to the unipotent radical of $\mathrm{GL}(2,\mathbb{F}_q)$. For H the split torus of $\mathrm{GL}(2,\mathbb{F}_q)$, we have that $Q_H^{\mathrm{GL}_2}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = q+1$ and $Q_H^{\mathrm{GL}_2}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) = 1$. Thus $Q_T^{C_i}(u) = 1$ for u not equal to the identity and $Q_T^{C_i}(I) = q+1$.

2.2. **Determination of unipotent parts of certain un.** Given an $n \times n$ matrix A, let I_n denote the $n \times n$ identity matrix and let $c(x) = \det(xI_n - A)$ denote the characteristic polynomial of A. Recall the basic linear algebra fact that there exist elementary row and column operations on the matrix $xI_n - A$ such that $xI_n - A$ reduces to

$$\begin{pmatrix} m_1(x) & 0 & 0 & 0 \\ 0 & m_2(x) & 0 & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & m_n(x) \end{pmatrix}$$

where the polynomial $m_i(x)$ divides $m_{i+1}(x)$ and $m_n(x)$ is the minimal polynomial of A. The polynomials $m_1(x), \ldots, m_n(x)$ are called the *invariant factors* of A. The following three lemmas will be used to determine the unipotent parts of un for $u \in U$ and $n = n_1, n_2$, and n_3. (For $n = n_0$, the unipotent part of un is clearly u and so the corresponding lemma when $n = n_0$ will not be needed.) The proofs of all three lemmas are similar, so only the proof of the first lemma will be provided.

Lemma 2.5. *Let $n = n_1$ and let $u = u(h, f, d, e)$. Let $c(x)$ and $m(x)$ denote the characteristic and minimal polynomials of un. Then $c(x) = m(x)$ except in the following situations:*
Situation 1) $d = e = 0$ and $h = f$. In this case the invariant factors of un are $1, 1, x^2 + fx + 1, x^2 + fx + 1$.
Situation 2) $d \neq 0, e = d, h = 2$, and $f = 2 - d^2$. In this case the invariant factors of un are $1, 1, x + 1, x^3 + (3 - d^2)x^2 + (3 - d^2)x + 1$.
Situation 3) $d \neq 0, e = -d, h = -2$, and $f = -2 + d^2$. In this case the invariant factors of un are $1, 1, x - 1, x^3 + (-3 + d^2)x^2 + (3 - d^2)x - 1$.

Proof of Lemma 2.5. We will perform a series of elementary row and column operations on $xI_n - un$:

$$xI_n - un = \begin{pmatrix} x+f & e & -d & -1 \\ e-dh & x+h & -1 & 0 \\ -d & 1 & x & 0 \\ 1 & 0 & 0 & x \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 0 & 0 & x \\ -d & 1 & x & 0 \\ x+f & e & -d & -1 \\ e-dh & x+h & -1 & 0 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & dx \\ 0 & e & -d & -x^2-fx-1 \\ 0 & x+h & -1 & (dh-e)x \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & dx \\ 0 & 0 & -x^2-hx-1 & -dx^2-ex \\ 0 & e & -d & -x^2-fx-1 \end{pmatrix}.$$

Note if $e = 0$ but $d \neq 0$ this last matrix, with a few more elementary operations, becomes

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c(x) \end{pmatrix}.$$

Whereas if $e = d = 0$ this matrix is
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & -x^2-hx-1 & 0 \\ 0 & 0 & 0 & -x^2-fx-1 \end{pmatrix}.$$
So if $e = d = 0$ and $f = h$ we have invariant factors $x^2 + fx + 1, x^2 + fx + 1$. However, if $e = d = 0$ and $f \neq h$ then we see that we again get the matrix C above.

Now suppose $e \neq 0$. Starting with the matrix following step 3 above, we perform a series of elementary operations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & dx \\ 0 & 0 & -x^2-hx-1 & -dx^2-ex \\ 0 & e & -d & -x^2-fx-1 \end{pmatrix} \xrightarrow{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & dx \\ 0 & 0 & -x^2-hx-1 & -dx^2-ex \\ 0 & 0 & -ex-d & -x^2-fx-1-edx \end{pmatrix} \xrightarrow{5} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-1}dx-hx-1 & -ex+e^{-1}x^3+e^{-1}fx^2+e^{-1}x \\ 0 & 0 & -ex-d & -x^2-fx-1-edx \end{pmatrix}.$$

In this last matrix, if we assume $d = h = 0$, we can see from the 2,3 entry that we can again reduce to the matrix C. However if $d = 0$ but $h \neq 0$ this matrix becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -hx-1 & -ex+e^{-1}x^3+e^{-1}fx^2+e^{-1}x \\ 0 & 0 & -ex & -x^2-fx-1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -ex+e^{-1}x^3+e^{-1}fx^2+e^{-1}x+e^{-1}hx^2+e^{-1}hfx+e^{-1}h \\ 0 & 0 & -ex & -x^2-fx-1 \end{pmatrix},$$

which again demonstrates we can reduce to C. So in all cases were $e \neq 0$ and $d = 0$ we have the matrix $xI_n - un$ reduces by elementary operations to C.

So for the remainder of this proof assume $e \neq 0$ and $d \neq 0$. Starting with the lower right hand two by two corner of the matrix after step 5 we have:

$$\begin{pmatrix} (e^{-1}d-h)x-1 & -ex+e^{-1}x^3+e^{-1}fx^2+e^{-1}x \\ -ex-d & -x^2-fx-1-edx \end{pmatrix} \xrightarrow{a} \begin{pmatrix} -hx-1-e^{-2}d^2 & -ex+e^{-1}x^3+e^{-1}fx^2+e^{-1}x-e^{-2}dx^2-e^{-2}dfx-e^{-2}d-e^{-1}d^2x \\ -ex-d & -x^2-fx-1-edx \end{pmatrix}.$$

Let $A(x)$ denote the polynomial in the upper right hand entry. Suppose $h = 0$. Then the $xI_n - un$ reduces to C if $-1-e^{-2}d^2 \neq 0$. So suppose $h = 0$ and $-1-e^{-2}d^2 = 0$. That is $d^2 = -e^2$. Then the matrix following step a becomes

$$\begin{pmatrix} 0 & A(x) \\ -ex-d & -x^2-fx-1-edx \end{pmatrix} \to \begin{pmatrix} 0 & A(x) \\ -ex-d & -fx-edx+e^{-1}dx-1 \end{pmatrix} \to \begin{pmatrix} 0 & A(x) \\ -ex-d & -fx+e^{-1}dx-1+d^2 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & A(x) \\ -ex-d & -fx-1+d^2-e^{-2}d^2 \end{pmatrix} = \begin{pmatrix} 0 & A(x) \\ -ex-d & -fx+d^2 \end{pmatrix}.$$

Notice that if $f = 0$, we get $xI_n - un$ reduces to C since we are already assuming $d \neq 0$. So assume $f \neq 0$. Then this last matrix becomes
$$\begin{pmatrix} 0 & A(x) \\ -ex-d & fe^{-1}d+d^2 \end{pmatrix}.$$
So we get $cI_n - un$ reduces to C except possibly when $fe^{-1}d + d^2 = 0$. That is when $f = -ed$. In conclusion, we have a possible case when $xI_n - un$ does not reduce to C when $e \neq 0, d \neq 0, d^2 = -e^2, h = 0$, and $f = -ed$. We will return to this case later.

The only remaining case is when d, e, and h are all nonzero. Returning to the matrix following step a above, but assuming now $h \neq 0$, we have

$$\begin{pmatrix} -hx-1-e^{-2}d^2 & A(x) \\ -ex-d & -x^2-fx-1-edx \end{pmatrix} \to \begin{pmatrix} -1-e^{-2}d^2+hde^{-1} & A(x)+he^{-1}x^2+he^{-1}fx+he^{-1}+hdx \\ -ex-d & -x^2-fx-1-edx \end{pmatrix}.$$

Let $B(x)$ denote the polynomial in the upper right hand entry. Note $xI_n - un$ reduces to C unless $-1 - e^{-2}d^2 + hde^{-1} = 0$, i.e., unless $h = d^{-1}e + de^{-1}$. Suppose $h = d^{-1}e + de^{-1}$. Then the above last matrix becomes

$$\begin{pmatrix} 0 & B(x) \\ -ex-d & -x^2-fx-1-edx \end{pmatrix} \to \begin{pmatrix} 0 & B(x) \\ -ex-d & -fx-edx+e^{-1}dx-1 \end{pmatrix} \to \begin{pmatrix} 0 & B(x) \\ -ex-d & -fx-edx-1-e^{-2}d^2 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & B(x) \\ -ex-d & -fx-1-e^{-2}d^2+d^2 \end{pmatrix}.$$

Thus, note if $f = 0$ we have $xI_n - un$ reduces to C except possibly when $-1 - e^{-2}d^2 + d^2 = 0$. That is, unless $-ed^{-1} - e^{-1}d + ed = 0$. That is unless $h = ed$. However if $f \neq 0$, this last matrix, after an elementary column operation, becomes

$$\begin{pmatrix} 0 & B(x) \\ -ex-d & -1-e^{-2}d^2+d^2+fe^{-1}d \end{pmatrix}.$$

Thus $xI_n - un$ reduces to C unless $-1-e^{-2}d^2+d^2+fe^{-1}d = 0$. That is unless $f = ed^{-1}+e^{-1}d-ed = h-ed$. (Note these conditions reduce to the previous conditions when $f = 0$.) In conclusion, we have a possible case where $xI_n - un$ does not reduce to C, when $e \neq 0, d \neq 0, h \neq 0, h = ed^{-1} + e^{-1}d$, and $f = h - ed$.

Thus there are two possible cases, when $e \neq 0$ and $d \neq 0$, where the matrix $xI_n - un$ may not reduce to C:

Case 1: $e \neq 0, d \neq 0, h = 0, d^2 = -e^2$, and $f = -ed$.

Case 2: $e \neq 0, d \neq 0, h \neq 0, h = ed^{-1} + e^{-1}d$, and $f = h - ed$.

We will now show that the matrix $xI_n - un$ always reduces to C in Case 1 but there are situations in Case 2 where $xI_n - un$ does not reduce to C.

Assume we are in Case 1. Then

$$xI - un = \begin{pmatrix} x-ed & e & -d & -1 \\ e & x & -1 & 0 \\ -d & 1 & x & 0 \\ 1 & 0 & 0 & x \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & dx \\ 0 & x & -1 & -ex \\ 0 & e & -d & -x^2+edx-1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -x^2-1 & -dx^2-ex \\ 0 & 0 & -ex-d & -x^2-1 \end{pmatrix}.$$

Taking the lower 2×2 corner matrix and interchanging rows and using $d^2 = -e^2$, we have

$$\begin{pmatrix} -ex-d & -x^2-1 \\ -x^2-1 & -dx^2-ex \end{pmatrix} \to \begin{pmatrix} -ex-d & -x^2-1 \\ e^{-1}dx-1 & e^{-1}x^3+e^{-1}x-dx^2-ex \end{pmatrix} \to \begin{pmatrix} -ex-d & -x^2-1 \\ 0 & e^{-1}x^3+e^{-1}x-dx^2-ex-e^{-2}dx^2-e^{-2}d \end{pmatrix}$$

$$\to \begin{pmatrix} cc-ex-d & -x^2-1 \\ 0 & x^3+x-edx^2-e^2x-e^{-1}dx^2-e^{-1}d \end{pmatrix} \to \begin{pmatrix} ccx+e^{-1}d & 0 \\ 0 & x^3+x-edx^2-e^2x-e^{-1}dx^2-e^{-1}d \end{pmatrix}.$$

Thus $xI_n - un$ does not reduce to C if and only if $x + e^{-1}d$ divides $x^3 + x - edx^2 - e^2x - e^{-1}dx^2 - e^{-1}d$. However, the remainder when $x^3 + x - edx^2 - e^2x - e^{-1}dx^2 - e^{-1}d$ is divided by $x + e^{-1}d$ is $2ed$, which is not zero (since e and d are nonzero). Thus in Case 1, $xI_n - un$ reduces to C.

Assume we are in Case 2. Then

$$xI_n - un = \begin{pmatrix} x+ed^{-1}+e^{-1}d-ed & e & -d & -1 \\ -e^{-1}d^2 & x+ed^{-1}+e^{-1}d & -1 & 0 \\ -d & 1 & x & 0 \\ 1 & 0 & 0 & x \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & dx \\ 0 & e & -d & -x^2-ed^{-1}x-e^{-1}dx+edx-1 \\ 0 & x+e^{-1}d+ed^{-1} & -1 & e^{-1}d^2x \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -ex-d & -x^2-ed^{-1}x-e^{-1}dx-1 \\ 0 & 0 & -x^2-ed^{-1}x-e^{-1}dx-1 & -dx^2-ex \end{pmatrix}$$

Taking the lower 2×2 corner matrix and continuing elementary operations we have

$$\begin{pmatrix} -ex-d & -x^2-ed^{-1}x-e^{-1}dx-1 \\ -ed^{-1}x-1 & e^{-1}x^3+d^{-1}x^2+e^{-2}dx^2-dx^2-ex+e^{-1}x \end{pmatrix} \to \begin{pmatrix} -ex-d & -x^2-ed^{-1}x-e^{-1}dx-1 \\ 0 & e^{-1}x^3+2d^{-1}x^2+e^{-2}dx^2-dx^2-ex+2e^{-1}x+ed^{-2}x+d^{-1} \end{pmatrix}$$

$$\to \begin{pmatrix} -ex-d & 0 \\ 0 & x^3+2ed^{-1}x^2+e^{-1}dx^2-edx^2-e^2x+2x+e^2d^{-2}x+ed^{-1} \end{pmatrix}.$$

Thus $xI_n - un$ will not reduce to C if and only if $x + e^{-1}d$ divides $x^3 + 2ed^{-1}x^2 + e^{-1}dx^2 - edx^2 - e^2x + 2x + e^2d^{-2}x + ed^{-1}$. The remainder when $x^3 + 2ed^{-1}x^2 + e^{-1}dx^2 - edx^2 - e^2x + 2x + e^2d^{-2}x + ed^{-1}$ is divided by $x + e^{-1}d$ is $ed - e^{-1}d^3$. Thus $xI_n - un$ will not reduce to C if and only if $ed - e^{-1}d^3 = 0$. That is, $e^2 = d^2$. So $e = \pm d$. Thus the only time we have that $xI_n - un$ does not reduce to C (when $e \neq 0$ and $d \neq 0$) is when $h \neq 0, h = ed^{-1} + e^{-1}d, f = h - ed$, and $e = \pm d$. If $e = d$, these conditions on h and f simplify to $h = 2$ and $f = 2 - d^2$. If $e = -d$, these conditions on h and f simplify to $h = -2$ and $f = -2 + d^2$. Finally, notice that the bottom right hand corner of this final matrix simplifies to $x^3 + (3 - d^2)x^2 + (3 - d^2)x + 1$ when $d = e$ and to $x^3 + (-3 + d^2)x^2 + (3 - d^2)x - 1$ when $d = -e$, giving us the invariant factors in the statement of the lemma. $\square$

Lemma 2.6. *Let $n = n_2$ and $u = u(h, f, d, e)$. Let $c(x)$ and $m(x)$ denote the characteristic and minimal polynomials of un. Then $c(x) = m(x)$ except in the following situations:*

Situation 1) $h = d = f = e = 0$. In this case the invariant factors of un are $1, 1, x^2 - 1$, and $x^2 - 1$.

Situation 2) $h = e = 0$, and $f = 2d \neq 0$. In this case the invariant factors of un are $1, 1, x - 1$, and $(x + 1)^2(x - 1)$.

Situation 3) $h = e = 0$, and $f = -2d \neq 0$. In this case the invariant factors of un are $1, 1, x + 1$, and $(x + 1)(x - 1)^2$.

Lemma 2.7. *Let $n = n_3$ and let $u = u(h, f, d, e)$. Let $c(x)$ and $m(x)$ denote the characteristic and minimal polynomials of un. Then $c(x) = m(x)$ except in the following situations:*

Situation 1) $d=e=h=0$, *and* $f=-2$. *In this case the invariant factors of* un *are* 1, $(x-1)$, $(x-1)$, *and* $(x-1)^2$.

Situation 2) $d=e=0$, $h\neq 0$, *and* $f=-2$. *In this case the invariant factors of* un *are* $1,1,(x-1)^2$, *and* $(x-1)^2$.

Situation 3) $d=e=h=0$, *and* $f\neq -2$. *In this case the invariant factors of* un *are* $1,1,(x-1)$, *and* $(x-1)(x^2+fx+1)$.

Situation 4) $d=0, e\neq 0, h\neq 0$, *and* $f=e^2h^{-1}-2$. *In this case the invariant factors of* un *are* $1,1,(x-1)$, *and* $(x-1)(x^2+fx+1)$.

2.3. **Green function values on unipotent parts of certain un.** In the following three lemmas, the values of the Green function on the unipotent parts of un whose semisimple part is a split torus element are provided for $n=n_1,n_2$, and n_3. (For $n=n_0$, the unipotent part of un is clearly u, so, in this case, the Green function values are listed in subsection 1.3.) In each lemma, only those values of t where there exist conditions on u such that there exists a $g\in G$ where $t=(gung^{-1})_s$ are listed. These conditions were determined in Lemmas 2.2, 2.3, and 2.4.

Lemma 2.8. *Let* $n=n_1$ *and let* $t\in T$. *The following chart lists the values of the Green functions* $Q_T^{C_G(t)}((un)_u)$ *for those* $u\in U$ *such that there exists* $g\in G$ *where* $(gung^{-1})_s=t$.

a. If $t=t(1,1,1)$ *then*

$$Q_T^{C_G(t)}((un)_u)=\begin{cases} 3q+1 & \text{if } d=e=0, f=h=-2\in(\mathbb{F}_q^\times)^2,\\ q+1 & \text{if } d=e=0, f=h=-2\notin(\mathbb{F}_q^\times)^2,\\ 1 & \text{otherwise.}\end{cases}$$

If $t=t(-1,-1,1)$ *then*

$$Q_T^{C_G(t)}((un)_u)=\begin{cases} 3q+1 & \text{if } d=e=0, f=h=2\in(\mathbb{F}_q^\times)^2,\\ q+1 & \text{if } d=e=0, f=h=2\notin(\mathbb{F}_q^\times)^2,\\ 1 & \text{otherwise.}\end{cases}$$

b. If $t=t(-1,1,1)$ *or* $t(1,-1,1)$ *then*

$$Q_T^{C_G(t)}((un)_u)=\begin{cases} q+1 & \text{if } d^2=4, e=d, h=2, f=-2,\\ q+1 & \text{if } d^2=4, e=-d, h=-2, f=2,\\ 1 & \text{otherwise.}\end{cases}$$

If $t = t(-i, i, -1)$ *or* $t(i, -i, -1)$ *then*

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } h = f = d = e = 0, \\ 1 & \text{otherwise.} \end{cases}$$

c. If $t = t(a, a, 1), (a^2 \neq 1)$ *then*

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } d = e = 0, h = f = -a - a^{-1}, \\ 1 & \text{otherwise.} \end{cases}$$

If $t = t(1, 1, -1)$ *or* $t(-1, -1, -1)$ *then*

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } d = e \neq 0, d^2 = 4, h = 2, f = -2, \\ q+1 & \text{if } d = -e \neq 0, d^2 = 4, h = -2, f = 2, \\ 1 & \text{otherwise.} \end{cases}$$

d. If $t = t(a, 1, 1)$ *and* $a^2 \neq 1$ *then*

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } e = -d \neq 0, f = -a - a^{-1}, h = -2, d^2 = 2 - a - a^{-1}, \\ 1 & \text{otherwise.} \end{cases}$$

If $t = t(a, -1, 1)$ *and* $a^2 \neq 1$ *then*

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } e = d \neq 0, f = -a - a^{-1}, h = 2, d^2 = 2 + a + a^{-1}, \\ 1 & \text{otherwise.} \end{cases}$$

e. If $t = t(a, b, \pm 1), a \neq b, a^2 \neq \pm 1, b^2 \neq \pm 1,$ *then* $Q_T^{C_G(t)}((un)_u) = 1$.

Proof of Lemma 2.8.

Since Green functions are equal to 1 at any regular element, the only places where the Green function is not 1 is possibly at Situations 1, 2, and 3 of Lemma 2.5. (A regular unipotent element is an element contained in the conjugacy class of unipotent elements with the smallest centralizer.)

First suppose $t = t(1, 1, 1)$. Then, by Lemma 2.2, $f = -h - 4$ and $e^2 - hde + d^2 + (h+2)^2 = 0$. If Situation 1 occurs we also need $d = e = 0$ and $f = h$. So we have $2f = -4$, i.e., $h = f = -2$. Also, $e^2 - hde + d^2 + (h+2)^2 = 0$ reduces to $0 = 0$. So Situation 1 does occur but only for $h = f = -2$. If Situation 2 occurs then we instead need $d \neq 0, e = d, h = 2$, and $f = 2 - d^2$. Then we would have $0 = e^2 - hde + d^2 + (h+2)^2 = d^2 - 2d^2 + d^2 + 4^2 = 16$. So Situation 2 does not occur. If Situation 3 occurs then we instead need $d \neq 0, e = -d, h = -2$, and $f = -2 + d^2$. Then $f = -(h+4) = -2$. So $f = -2 + d^2$ becomes $-2 = -2 + d^2$. So we would have $d^2 = 0$, which is a contradiction. Therefore, Situation 3 does not occur. Thus, when $t = t(1,1,1)$ only Situation 1 occurs. By Lemma 2.5, the invariant factors of un in this case are $1, 1, x^2 + fx + 1, x^2 + fx + 1$. Also $C_G(t) = G$. Thus $Q_T^{C_G(t)}(un)_u) = Q_T^G(A_{31})$ or $Q_T^G(A_{32})$ depending on whether or not f is in $(\mathbb{F}_q^\times)^2$. This completes the proof in the case $t = t(1,1,1)$.

Now suppose $t = t(-1,-1,1)$. Then, by Lemma 2.2, $f = -h + 4$ and $e^2 - hde + d^2 + (h-2)^2 = 0$. If Situation 1 occurs, we also need $d = e = 0$ and $f = h$. So we have $2f = 4$, i.e., $h = f = 2$. Also, $e^2 - hde + d^2 + (h-2)^2 = 0$ just reduces to $0 = 0$. So Situation 1 does occur but only for $h = f = 2$. If Situation 2 occurs then we instead need $d \neq 0, e = d, h = 2$, and $f = 2 - d^2$. Then we would have $f = -h + 4 = -2 + 4 = 2$. So $f = 2 - d^2$ becomes $2 = 2 - d^2$. That is, $d^2 = 0$, which is a contradiction. So Situation 2 does not occur. If Situation 3 occurs then we instead need $d \neq 0, e = -d, h = -2$, and $f = -2 + d^2$. Then we would have $0 = e^2 - hde + d^2 + (h-2)^2 = d^2 - 2d^2 + d^2 + (-4)^2 = 16$. So Situation 3 does not occur. Thus, when $t = t(-1,-1,1)$ only Situation 1 occurs. By Lemma 2.7, the invariant factors of un in this case are $1, 1, x^2 + fx + 1, x^2 + fx + 1$. Also $C_G(t) = G$. Thus $Q_T^{C_G(t)}((un)_u) = Q_T^G(A_{31})$ or $Q_T^G(A_{32})$ depending on whether or not f is in $(\mathbb{F}_q^\times)^2$.

Note that for $t \in T$ that lie in Case b, c or d, we have that $C_G(t) = C_i$, for some $i = 1, 2$, or 3. Thus, in these cases, $Q_T^{C_G(t)}((un)_u) = Q_T^{C_i}((un)_u)$ will be equal to $q+1$ precisely when the conditions on u are such that we are in Situations 1, 2 or 3 of Lemma 2.5. Otherwise $Q_T^{C_i}((un)_u) = 1$.

Now suppose t has the form of Case b in Lemma 2.2. By Lemma 2.2, we have $h = -f$ and $e^2 - hde + d^2 + h^2 - 2a^2 - 2 = 0$. If Situation 1 occurs we also have $d = e = 0$ and $h = f$. So $d = e = f = h = 0$. Thus the condition $e^2 - hde + d^2 + h^2 - 2a^2 - 2 = 0$ becomes $a^2 = -1$. So $a = \pm i$. If Situation 2 then we instead need $d \neq 0, e = d, h = 2$, and $f = 2 - d^2$. Thus $f = -2$ and $f = 2 - d^2$ becomes $d^2 = 4$. For $d = \pm 2$, the equation $0 = e^2 - hde + d^2 + h^2 - 2a^2 - 2$ simplifies to $a^2 = 1$. So Situation 2 occurs precisely when $a = \pm 1, e = d, h = 2, f = -2$, and $d^2 = 4$. For Situation 3 we instead need $d \neq 0, e = -d, h = -2$, and $f = -2 + d^2$. So $f = 2$. Then $f = -2 + d^2$ simplifies again to $d^2 = 4$ and $0 = e^2 - hde + d^2 + h^2 - 2a^2 - 2$ again simplifies to $a^2 = 1$. So Situation 3 occurs precisely when $a = \pm 1, e = -d, h = -2, f = 2$, and $d^2 = 4$.

Now suppose t has the form of Case c in Lemma 2.2. Suppose Situation 1 occurs. Then $d = e = 0$ and $h = f$. If $c = -1$, we have $f = -h$ which forces $f = h = 0$. We also have $e^2 - hde + d^2 + h^2 - 4 = 0$ which now simplifies to $-4 = 0$. So Situation 1 does not occur when $c = -1$. So we assume $c = 1$. Then we have the semisimple condition $e^2 - hde + d^2 + \frac{(a^2+ah+1)^2}{a^2} = 0$, which simplifies to $h = -a - a^{-1}$. We also have the semisimple condition $f = -2a - h - 2a^{-1}$ which, using $f = h$, also simplifies to $f = -a - a^{-1}$. So Situation 1 occurs precisely when $c = 1, d = e = 0$, and $f = h = -a - a^{-1}$. Suppose Situation 2 occurs and $c = -1$. Then $h = 2$ and $e = d$. Then, by the semisimple conditions, $f = -h$ and so $f = -2$. However, we also have

$f = 2 - d^2$. So $d^2 = 4$. Note that the condition $e^2 - hde + d^2 + h^2 - 4 = 0$ is satisfied. Thus, when $c = -1$, Situation 2 is precisely satisfied when $a^2 = 1, d = e, d^2 = 4, h = 2$, and $f = -2$. If Situation 3 occurs when $c = -1$, we get $h = -2, e = -d$, and $-2 + d^2 = f$. Thus, using $f = -h$, we have $f = 2$, and $d^2 = 4$. Thus, when $c = -1$, Situation 3 is precisely satisfied when $a^2 = 1, d = -e, d^2 = 4, h = -2$, and $f = 2$. Suppose Situation 2 occurs when $c = 1$. We have $h = 2, e = d$, and$f = 2 - d^2$. Then the semisimple condition $e^2 - hde + d^2 + \frac{(a^2+ah+1)^2}{a^2} = 0$ becomes $d^2 - 2d^2 + d^2 + \frac{(a^2+2a+1)^2}{a^2} = 0$. However, the only solution to this equation is when $a = -1$, so the semisimple condition $f = -2a - h - 2a^{-1}$ becomes $f = 2 - 2 + 2 = 2$. We would also have $f = 2 - d^2$, which is $2 = 2 - d^2$. So $d = 0$, which is a contradiction. Thus, Situation 2 does not occur when $c = 1$. Suppose Situation 3 occurs when $c = 1$. We have $h = -2, e = -d$, and$f = -2 + d^2$. Then the semisimple condition $e^2 - hde + d^2 + \frac{(a^2+ah+1)^2}{a^2} = 0$ becomes $d^2 - 2d^2 + d^2 + \frac{(a^2-2a+1)^2}{a^2} = 0$. The only solution to this equation is when $a = 1$. Then the semisimple condition $f = -2a - h - 2a^{-1}$ becomes $f = -2 + 2 - 2 = -2$. However, we also have $f = -2 + d^2$, i.e., $-2 = -2 + d^2$. So $d = 0$, which is a contradiction. Thus, Situation 3 does not occur when $c = 1$.

Now suppose t has the form of Case d in Lemma 2.2. Suppose Situation 1 occurs. Then $d = e = 0$ and $h = f$. We have the semisimple conditions $a^2 \neq b^2, b^2 = 1, f = -a - h - 2b - a^{-1}$, and $e^2 - hde + d^2 + \frac{(h+2b)(a^2+ah+1)}{a} = 0$. Since $d = e = 0$, we have $(h + 2b)(a^2 + ah + 1) = 0$. So $h = -2b$ or $h = -a - a^{-1}$. Suppose $h = -2b$. Then, since $h = f$ and $f = -a - h - 2b - a^{-1}$, we have $-2b = -a + 2b - 2b - a^{-1}$. So $-2b = -a-a^{-1}$. However, $b = \pm 1$. So $\pm 2 = -a-a^{-1}$ and thus $0 = a^2 \pm 2a+1$. That is, $a = \pm 1$, contradicting the condition $a^2 \neq b^2$. So $h \neq -2b$. Thus, $h = -a - a^{-1}$. Since $h = f$ and $f = -a - h - 2b - a^{-1}$, we have $-a - a^{-1} = -a + a + a^{-1} - 2b - a^{-1}$, which again gives us $-2b = -a - a^{-1}$. This leads to the contradiction $a = \pm 1$. So $h \neq -a-a^{-1}$. Thus Situation 1 does not occur. Now consider Situation 2. Since $e = d$ and $h = 2$, the semisimple condition $e^2 - hde + d^2 + \frac{(h+2b)(a^2+ah+1)}{a} = 0$ tells us $(2+2b)(a^2+2a+1) = 0$. Thus $b = -1$ or $a = -1$. Since $b^2 = 1$ and $a^2 \neq b^2$, we have $a \neq -1$. Thus, $b = -1$ and Situation 2 does not occur for $b = 1$. For $b = -1$, the semisimple condition $f = -a-h-2b-a^{-1}$ becomes $f = -a-a^{-1}$ since $h = 2$. We also have $f = 2 - d^2$. Thus, $d^2 = 2 + a + a^{-1}$. So Situation 2 occurs only when $b = -1, d = e, h = 2, f = -a - a^{-1}$, and $d^2 = 2 + a + a^{-1}$. Now consider Situation 3. Since $e = -d$ and $h = -2$, the semisimple condition $e^2 - hde + d^2 + \frac{(h+2b)(a^2+ah+1)}{a} = 0$ tells us $(-2 + 2b)(a^2 - 2a + 1) = 0$. So $b = 1$ or $a = 1$. Since we have $b^2 = 1$ and $a^2 \neq b^2$, then we must have $a \neq 1$. So $b = 1$. Thus, Situation 3 does not occur for $b = -1$. For $b = 1$, the semisimple condition $f = -a - h - 2b - a^{-1}$ becomes $f = -a - a^{-1}$ since $h = -2$. We also have $f = -2 + d^2$. So $d^2 = 2 - a - a^{-1}$. Thus, Situation 3 occurs only when $b = 1, d = -e, h = -2, f = -a - a^{-1}$, and $d^2 = 2 - a - a^{-1}$.

Finally, suppose t has the form of Case e in Lemma 2.2. Then $C_G(t) = T$ and so the Green function $Q_T^{C_G(t)} = Q_T^T$ is identically 1. □

Lemma 2.9. *Let $n = n_2$ and let $t \in T$. The following chart lists the values of the Green functions $Q_T^{C_G(t)}((un)_u)$ for those u such that $(gung^{-1})_s = t$ for some $g \in G$.*

a. If $t = t(1,1,1)$ or $t(-1,-1,1)$, then $Q_T^{C_G(t)}((un)_u) = 1$.

b. If $t = t(-1,1,1), t(1,-1,1)$ then

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } h = f = d = e = 0, \\ q+1 & \text{if } h = e = 0, f = \pm 2d \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

If $t = t(-i,i,-1)$ or $t(i,-i,-1)$ then $Q_T^{C_G(t)}((un)_u) = 1$.

c. If $t = t(a,a,1), (a^2 \neq 1)$ then $Q_T^{C_G(t)}((un)_u) = 1$

If $t = t(1,1,-1)$ or $t(-1,-1,-1)$, then

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } h = f = d = e = 0, \\ q+1 & \text{if } h = e = 0, f = \pm 2d \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

d. If $t = t(a, \pm 1, 1)$, $a^2 \neq 1$ then $Q_T^{C_G(t)}((un)_u) = 1$

e. If $t = t(a,b,\pm 1), a \neq b, a^2 \neq \pm 1, b^2 \neq \pm 1$ then $Q_T^{C_G(t)}((un)_u) = 1$.

Proof of Lemma 2.9.

Since Green functions are equal to 1 at any regular element, the only places where the Green function is not 1 are possibly at Situations 1, 2, and 3 of Lemma 2.6. Thus, by Lemma 2.6, $(un)_u$ will be a regular element if either h or e is nonzero.

In case a of Lemma 2.3, we have the semisimple condition $dh = \pm 4$ which forces h to be nonzero. In case d the condition $d = \frac{(a+b)^2}{ah}$ forces h to be nonzero. Thus, the Green functions in cases a and d are identically 1. In case e we have $C_G(t) = T$. Thus, the Green function $Q_T^{C_G(t)}$ is again one. This leaves cases b and c. Assume $h = e = 0$ (since otherwise $(un)_u$ is regular). For case b, we have $a^4 = 1, hd = 0$, and $0 = e^2 - fh - 2a^2 + 2 = -2a^2 + 2$. Thus $a^2 = 1$. This is compatible with all three situations of Lemma 2.6 when $a = \pm 1$ otherwise the Green function is 1 in case b. For case c when $t = (a,a,1)$ we have $e^2 + 4 = 0$ by the semisimple conditions forcing $e \neq 0$ and thus $(un)_u$ is regular. For case c when $t = (a,a,-1)$ and $h = 0$ then $a^2 = 1$ and $e = 0$ according to the semisimple conditions. This is compatible with the situations in Lemma 2.6 for $t = (1,1,-1)$ and $t = (-1,-1,-1)$. $\square$

Lemma 2.10. *Let $n = n_3$ and let $t \in T$. The following chart lists the values of the Green functions $Q_T^{C_G(t)}((un)_u)$ for those u such that $(gung^{-1})_s = t$ for some $g \in G$.*

a. If $t = t(1,1,1)$, then

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} (q+1)^2 & \text{if } f=-2, h=d=e=0, \\ 3q+1 & \text{if } f=-2, d=e=0, h \in (\mathbb{F}_q^\times)^2, \\ q+1 & \text{if } f=-2, d=e=0, h \notin (\mathbb{F}_q^\times)^2, h \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

If $t = t(-1,-1,1)$, then $Q_T^{C_G(t)}((un)_u) = 1$.

b. If $t = t(-1,1,1)$ or $t(1,-1,1)$ then

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } f=2, h=d=e=0, \\ q+1 & \text{if } f=2, d=0, e^2=4h \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

If $t = t(-i,i,-1)$ or $t(i,-i,-1)$ then $Q_T^{C_G(t)}((un)_u) = 1$.

c. If $t = t(a,a,1)$, $(a^2 \neq 1)$ then $Q_T^{C_G(t)}((un)_u) = 1$.

If $t = t(1,1,-1)$ or $t(-1,-1,-1)$, then

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } f=2, h=d=e=0 \\ q+1 & \text{if } f=2, d=0, e^2=4h \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

d. If $t = t(a,1,1), (a^2 \neq 1)$ then

$$Q_T^{C_G(t)}((un)_u) = \begin{cases} q+1 & \text{if } h=d=e=0, f \neq -2 \\ q+1 & \text{if } d=0, h \neq 0, h(-a^{-1}-a+2)=e^2, f=-a-a^{-1} \\ 1 & \text{otherwise.} \end{cases}$$

If $t = t(a,-1,1), (a^2 \neq 1)$ then $Q_T^{C_G(t)}((un)_u) = 1$

e. If $t = t(a,b,\pm 1), a \neq b, a^2 \neq \pm 1, b^2 \neq \pm 1$ then $Q_T^{C_G(t)}((un)_u) = 1$.

Proof of Lemma 2.10. Suppose $t = t(1,1,1)$. Then, by the conditions for when $(gung^{-1})_s = t$ for some $g \in G$ given in Lemma 2.4, we have $f = -2$ and $hd = 0$. For Situation 1 in Lemma 2.7 to hold we also need $d = e = h = 0$. These sets of conditions are compatible so Situation 1 occurs and the invariant factors of $(un)_u$ in this case are $1, x-1, x-1$, and $(x-1)^2$. Thus $(un)_u$ is in the A_2 conjugacy class. Thus for this t and these conditions on u we have $Q_T^{C_G(t)}((un)_u) = Q_T^G(A_2) = (q+1)^2$. For Situation 2 in Lemma 2.7 to hold, we instead need $d = e = 0$ but $h \neq 0$. In this case we get $(un)_u$ has invariant factors $1, 1, x-1, (x-1)^2$. Thus $Q_T^{C_G(t)}((un)_u) = 3q+1$ when h is a nonzero square and $Q_T^{C_G(t)}((un)_u) = q+1$ when h is not a square. Note Situation 3 cannot occur since $f = -2$. For Situation 4 to hold we need $d = 0, e \neq 0, h \neq 0$, and $f + 2 = e^2 h^{-1}$. But $f = -2$, so $f + 2 = 0$. So $f + 2$ cannot equal $e^2 h^{-1}$ as both e and h are nonzero. So Situation 4 also does not occur in this case.

Suppose $t = t(-1, -1, 1)$. Then, by the semisimple conditions given in Lemma 2.4, we have $f = 6$ and $hd^2 = -16$. Thus $d \neq 0$. Thus, by Lemma 2.7, $(un)_u$ is regular and so $Q_T^{C_G(t)}((un)_u)$ is identically 1.

Note that for $t \in T$ that lie in Case b, c or d of this lemma, we have that the $C_G(t) = C_i$, for some $i = 1, 2$, or 3. Thus, in these cases, $Q_T^{C_G(t)}((un)_u) = Q_T^{C_i}((un)_u)$ will be equal to $q+1$ precisely when the conditions on u are such that we are in Situations 1, 2, 3 or 4 of Lemma 2.7. Otherwise $Q_T^{C_i}((un)_u) = 1$.

Suppose $t = t(-1, 1, 1)$ or $t(1, -1, 1)$. Then, by the semisimple conditions, we need $f = 2$ and $0 = hd^2 - 2a^2 + 2 = hd^2$. Since $f \neq -2$, Situations 1 and 2 of Lemma 2.7 cannot occur. Note both Situations 3 and 4 can occur and the restriction $f = e^2 h^{-1} - 2$ of Situation 4 simplifies to $4h = e^2$.

Suppose $t = t(-i, i, -1)$ or $t(i, -i, -1)$. Then, by the semisimple conditions, $f = 2$ and $0 = hd^2 - 2a^2 + 2 = hd^2 + 4$. Thus $d \neq 0$. So Situations 1–4 cannot occur.

Suppose $t = t(a, a, 1)$ with $a^2 \neq 1$. Then, by one of the semisimple conditions, $hd^2 + \frac{(a-1)^4}{a^2} = 0$. Suppose $d = 0$. This implies $(a-1)^4 = 0$ so that $a = 1$. However, we have $a^2 \neq 1$. So we must have $d \neq 0$. Thus, Situations 1–4 cannot occur when $t = t(a, a, 1)$ with $a^2 \neq 1$.

Now suppose $t = t(1, 1, -1)$ or $t(-1, -1, -1)$. Then, by the semisimple conditions, $f = 2$ and $hd = 0$. Thus Situations 1 and 2 in Lemma 2.7 cannot occur. Note both Situations 3 and 4 can occur and the restriction $f = e^2 h^{-1} - 2$ of Situation 4 again simplifies to $4h = e^2$.

Suppose $t = t(a, 1, 1)$ with $a^2 \neq 1$. Then, by the semisimple conditions, $f = -a - a^{-1}$ and $hd = 0$. Note that we cannot have $f = -2$ for if this were the case then $-2 = -a - a^{-1}$, which solving for a gives $a = 1$. So $f \neq -2$ and thus Situations 1 and 2 cannot occur. Note Situations 3 and 4 can occur. In Situation 4, since $f = -a - a^{-1}$ and $f = e^2 h^{-1} - 2$, we have $2 - a - a^{-1} = e^2 h^{-1}$.

Suppose $t = t(a, -1, 1)$ with $a^2 \neq 1$. Then, by the semisimple conditions, $f = -a - a^{-1} + 4$ and $hd^2 - \frac{4(a-1)^2}{a} = 0$. Suppose $d = 0$. This implies $(a-1)^2 = 0$ so that $a = 1$. However, we have $a^2 \neq 1$. So we must have $d \neq 0$. Thus Situations 1-4 cannot occur when $t = t(a, -1, 1)$ with $a^2 \neq 1$.

Finally consider $t = t(a, b, \pm 1)$ where $a \neq b$, $b^2 \neq \pm 1$, and $a^2 \neq \pm 1$. Then $C_G(t) = T$ and thus $Q_T^{C_G(t)} = Q_T^T$, which is always 1. $\square$

3. The irreducible representations of $\mathcal{H}$

Now that the appropriate values of Green functions have been determined, we can explicitly compute the coefficients $f_T(c_n)(t)$ of Curtis' map. This involves computing character sums. It turned out that explicit formulas for many of these sums could be found either by using elementary methods or methods from algebraic geometry. However, convenient explicit formulas could not be determined for the remaining sums. At best, upper bounds for such sums were found to exist in the literature, see Weil's Theorem [6, pg 43] for example.

Before we state our main results on the coefficients in Curtis' map, we remark that the coefficients $f_T(c_n)(t)$ for torus elements t whose multiplier is not ± 1 are 0 by Lemmas 2.2–2.4. The techniques that were used to prove Theorems 3.2–3.4 are provided following the statement of Theorem 3.4.

Theorem 3.1. *Let $n = n_o$. Then we have $f_T(c_n)(t) = 0$ for $t \neq t(1,1,1)$ and $f_T(c_n)(t(1,1,1)) = 1$.*

Proof. By Lemma 2.1, $f_T(c_n)(t) = 0$ for $t \neq t(1,1,1)$. So assume $t = t(1,1,1)$. Then

$$f_T(c_n)(t) = \frac{\operatorname{ind} n}{< Q_T^G, \Gamma > |U|\,|\mathrm{C}_G(t)|} \sum_{\substack{g \in G,\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{\mathrm{C}_G(t)}((gung^{-1})_u) = \frac{1}{q^4} \sum_{u \in U} \psi(u^{-1}) Q_T^G(u).$$

This last expression is the inner product of the characters ψ and Q_T^G and is thus equal to 1. □

Theorem 3.2. *The coefficients of the map in Curtis' theorem for the torus elements of multiplier ± 1 for $n = n_1$ are as follows.*

a. $t = (a, a, 1), a^2 = 1$.

Let C_{1a} be the set of solutions (h, d, e) to the equation

$$e^2 - hde + d^2 + (h + 2a)^2 = 0.$$

Then

$$f_T(c_n)(t) = \psi_1(2a) \cdot \begin{cases} 3q & \text{if } -2a \in (\mathbb{F}_q^\times)^2, \\ q & \text{if } -2a \notin (\mathbb{F}_q^\times)^2, \end{cases} + \sum_{\substack{(h,d,e) \in C_{1a} \\ h \neq -2a}} \psi_1(-h)\psi_2(-d).$$

b. $t = (-a, a, a^2), a^4 = 1$.

Let C_{1b+} be the set of solutions (h, d, e) to the equation

$$e^2 - hde + d^2 + h^2 - 4 = 0.$$

If $a^2 = 1$*, then*

$$f_T(c_n)(t) = (\psi_1(2) + \psi_1(-2)) \left[(q+1)(\psi_2(2) + \psi_2(-2)) + \sum_{d \in \mathbb{F}_q/\{\pm 2\}} \psi_2(d) \right]$$

$$+ \sum_{\substack{(h,d,e) \in C_{1b+}, \\ h^2 \neq 4}} \psi_1(-h)\psi_2(-d).$$

Let C_{1b-} *be the set of solutions* (h,d,e) *to the equation*

$$e^2 - hde + d^2 + h^2 = 0.$$

If $a^2 = -1$*, then*

$$f_T(c_n)(t) = q + 1 + \sum_{\substack{(h,d,e) \in C_{1b-} \\ (h,d,e) \neq (0,0,0)}} \psi_1(-h)\psi_2(-d).$$

c. $t = (a,a,c), c \neq a^2, c^2 = 1.$

Let C_{1c+} *be the set of solutions* (h,d,e) *to the equation*

$$e^2 - hde + d^2 + \frac{(a^2 + ah + 1)^2}{a^2} = 0.$$

If $c = 1$*, then*

$$f_T(c_n)(t) = \psi_1(-a^{-1} - a)(q+1) + \sum_{\substack{(h,d,e) \in C_{1c+} \\ (d,e) \neq (0,0)}} \psi_1(-h)\psi_2(-d).$$

Let C_{1c-} *be the set of solutions* (h,d,e) *to the equation*

$$e^2 - hde + d^2 + h^2 - 4 = 0.$$

If $c = -1$*, then*

$$f_T(c_n)(t) = (q+1)(\psi_1(2) + \psi_1(-2))(\psi_2(2) + \psi_2(-2)) + \sum_{\substack{(h,d,e) \in C_{1c-} \\ (h,d,e) \neq (2,\pm 2,\pm 2)}} \psi_1(-h)\psi_2(-d).$$

d. $t = (a,b,1), 1 = b^2 \neq a^2.$

Let C_{1d} *be the set of solutions* (h,d,e) *to the equation*

$$e^2 - hde + d^2 + \frac{(h+2b)(a^2 + ah + 1)}{a} = 0.$$

Then

$$f_T(c_n)(t) = -\psi_1(2)\cdot\begin{cases}\frac{1+(-1)^{(q-1)/2}}{2}q+1 & \text{if } a\in(\mathbb{F}_q^\times)^2,\\ \frac{1-(-1)^{(q-1)/2}}{2}q+1 & \text{if } a\notin(\mathbb{F}_q^\times)^2,\end{cases} + \sum_{\substack{(h,d,e)\in C_{1d}\\ h\neq -2}} \psi_1(-h)\psi_2(-d).$$

e. $t=(a,b,c), a\neq b, c\neq b^2, c^2=1.$

Let C_{1e+} *be set of solutions* (h,d,e) *to the equation*

$$e^2 - hde + d^2 + h^2 + \frac{(a+b)(1+ab)h}{ab} + \frac{(a^2+1)(b^2+1)}{ab} = 0.$$

If $c=1$, *then*

$$f_T(c_n)(t) = \sum_{(h,d,e)\in C_{1e+}} \psi_1(-h)\psi_2(-d).$$

Let C_{1e-} *be set of solutions* (h,d,e) *to the equation*

$$e^2 - hde + d^2 + h^2 - \frac{(a^2+1)^2}{a^2} = 0.$$

If $c=-1$, *then*

$$f_T(c_n)(t) = \sum_{(h,d,e)\in C_{1e-}} \psi_1(-h)\psi_2(-d).$$

Theorem 3.3. *The coefficients of the map in Curtis' theorem for the torus elements of multiplier* ±1 *for* $n=n_2$ *are as follows.*

a. $t=(a,a,1), a^2=1.$

$$f_T(c_n)(t) = \sum_{h\in\mathbb{F}_q^\times} \psi_1(h)\psi_2\left(\frac{4a}{h}\right).$$

b. $t=(-a,a,a^2), a^4=1.$

If $a^2=1$, *then* $f_T(c_n)(t)=-2.$

If $a^2=-1$, *then* $f_T(c_n)(t)=-1.$

c. $t=(a,a,c), c\neq a^2, c^2=1.$

If $c=1$, *then*

$$f_T(c_n)(t) = \sum_{h\in\mathbb{F}_q^\times} \psi_1(h)\psi_2\left(\frac{2(a^2+1)}{ah}\right).$$

If $c=-1$, *then* $f_T(c_n)(t) = \dfrac{1-2q}{q}.$

d. $t = (a, b, 1), 1 = b^2 \neq a^2$.

$$f_T(c_n)(t) = \sum_{h \in \mathbb{F}_q^\times} \psi_1(h)\psi_2\left(-\frac{(a+b)^2}{ah}\right).$$

e. $t = (a, b, c), a \neq b, c \neq b^2, c^2 = 1$.

$$f_T(c_n)(t) = \sum_{h \in \mathbb{F}_q^\times} \psi_1(h)\psi_2\left(-\frac{(a+b)(ab+c)}{abh}\right).$$

Theorem 3.4. *The coefficients of the map in Curtis' theorem for the torus elements of multiplier* ± 1 *for* $n = n_3$ *are as follows.*

a. $t = (a, a, 1), a^2 = 1$.

If $a = 1$*, then*

$$f_T(c_n)(t) = q + 2 \sum_{h \in (\mathbb{F}_q^\times)^2} \psi_1(-h).$$

If $a = -1$*, then*

$$f_T(c_n)(t) = \sum_{d \in \mathbb{F}_q^\times} \psi_1\left(\frac{16}{d^2}\right)\psi_2(d).$$

b. $t = (-a, a, a^2), a^4 = 1$.

If $a^2 = 1$*, then*

$$f_T(c_n)(t) = \frac{2}{q} + \frac{2q+2}{q} \sum_{h \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2} \psi_1(-h).$$

If $a^2 = -1$*, then*

$$f_T(c_n)(t) = \sum_{d \in \mathbb{F}_q^\times} \psi_1\left(\frac{4}{d^2}\right)\psi_2(d).$$

c. $t = (a, a, c), c \neq a^2, c^2 = 1$.

If $c = 1$*, then*

$$f_T(c_n)(t) = \sum_{d \in \mathbb{F}_q^\times} \psi_1\left(\frac{(a-1)^4}{a^2 d^2}\right)\psi_2(d).$$

If $c = -1$*, then*

$$f_T(c_n)(t) = \frac{1}{q}\left((q+1)\sum_{h \in \mathbb{F}_q^\times} \psi_1\left(-\frac{h^2}{4}\right) + \sum_{\substack{h \in \mathbb{F}_q^\times, e \in \mathbb{F}_q^\times, \\ h \neq e^2/4}} \psi_1(-h)\right).$$

d. $t = (a, b, 1), 1 = b^2 \neq a^2$.

If $b = 1$*, then*

$$f_T(c_n)(t) = 2 \sum_{h : -\frac{h}{a} \in (\mathbb{F}_q^\times)^2} \psi_1(-h).$$

If $b = -1$, *then*

$$f_T(c_n)(t) = q \sum_{d \in \mathbb{F}_q^\times} \psi_1 \left(-\frac{4(a-1)^2}{ad^2} \right) \psi_2(d).$$

e. $t = (a,b,c), a \neq b, c \neq b^2, c^2 = 1.$

If $c = 1$, *then*

$$f_T(c_n)(t) = \sum_{d \in \mathbb{F}_q^\times} \psi_1 \left(\frac{(a-1)^2(b-1)^2}{abd^2} \right) \psi_2(d).$$

If $c = -1$, *then*

$$f_T(c_n)(t) = \sum_{d \in \mathbb{F}_q^\times} \psi_1 \left(-\frac{(a^2-1)^2}{a^2d^2} \right) \psi_2(d).$$

We omit the proofs. As an example of the steps taken to determine the coefficients, we include the computations in situation b in the case $n = n_2$ where $a^2 = 1$. The proofs of the other cases are similar.

If $a^2 = 1$, then $hd = 0$ and $e^2 - hf = 0$.

$$\begin{aligned}
f_T(c_n)(t) &= \frac{\operatorname{ind} n}{< Q_T^G, \Gamma > |U|\,|\mathrm{C}_G(t)|} \sum_{\substack{g \in G,\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{\mathrm{C}_G(t)}((gung^{-1})_u) \\
&= \frac{q^3}{q^4\,|\mathrm{C}_G(t)|} \sum_{\substack{g \in G,\ u \in U \\ (gung^{-1})_s = t}} \psi(u^{-1}) Q_T^{\mathrm{C}_G(t)}((gung^{-1})_u) \\
&= \frac{1}{q}\Bigg(\psi(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{-1})(q+1) + \sum_{f \in \mathbb{F}_q^\times} \psi(\begin{pmatrix} 1 & & & f \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{-1}) \cdot 1 + \sum_{d \in \mathbb{F}_q^\times, f^2 = 4d^2} \psi(\begin{pmatrix} 1 & d & & f \\ & 1 & & \\ & & 1 & d \\ & & & 1 \end{pmatrix}^{-1}) \cdot (q+1) \\
&\qquad + \sum_{d \in \mathbb{F}_q^\times, f^2 \neq 4d^2} \psi(\begin{pmatrix} 1 & d & & f \\ & 1 & & \\ & & 1 & -d \\ & & & 1 \end{pmatrix}^{-1}) \cdot 1 + \sum_{h \in \mathbb{F}_q^\times, e \in \mathbb{F}_q} \psi(\begin{pmatrix} 1 & & e & e^2/h \\ & 1 & h & e \\ & & 1 & \\ & & & 1 \end{pmatrix}^{-1}) \cdot 1 \Bigg) \\
&= \frac{1}{q}\Bigg(\psi\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}(q+1) + \sum_{f \in \mathbb{F}_q^\times} \psi\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + 2\sum_{d \in \mathbb{F}_q^\times} \psi\begin{pmatrix} 1 & d & & \\ & 1 & & \\ & & 1 & -d \\ & & & 1 \end{pmatrix} \cdot (q+1) \\
&\qquad + (q-2)\sum_{d \in \mathbb{F}_q^\times} \psi\begin{pmatrix} 1 & d & & \\ & 1 & & \\ & & 1 & -d \\ & & & 1 \end{pmatrix} + q\sum_{h \in \mathbb{F}_q^\times} \psi\begin{pmatrix} 1 & & & \\ & 1 & h & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Bigg) \\
&= \frac{1}{q}(q+1+q-1-2(q+1)-(q-2)-q) = -2.
\end{aligned}$$

As we have said before, elementary methods could not be used to find formulas for every sum. So other techniques had to be used. We now illustrate how the character value $f_T(c_{n_3})$ can be determined using results from algebraic geometry[1].

[1]The authors would like to thank Alex Mueller for explaining how to use zeta functions of curves to compute character sums.

We consider case (a) from Theorem 3.4, where $t = (1, 1, 1)$. Then

$$f_T(c_n)(t) = q + 2 \sum_{h \in (\mathbb{F}_q^\times)^2} \psi_1(-h).$$

Note that

$$2 \sum_{h \in (\mathbb{F}_q^\times)^2} \psi_1(-h) = \sum_{h \in \mathbb{F}_q^\times} \psi_1(-h^2).$$

Also, we have that $\psi_1(h) = \psi_0(\mathrm{Tr}(ah))$ for some character ψ_0 of $\mathbb{F}_p$ and some $a \in \mathbb{F}_q$.

Let X be the Artin-Schreier curve

$$X : y^q - y = -ah^2.$$

For any character ψ_0 of $\mathbb{F}_p$, we define an L-function of ψ_0 by

$$L(\psi_0, t) = \sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{h \in \mathbb{F}_{p^n}} \psi_0(\mathrm{Tr}_n(-ah^2)).$$

These are rational functions of t. The zeta function of X satisfies

$$Z(X, t) = \frac{\prod_{\psi_0 \neq 1} L(\psi_0, t)}{(1-t)(1-qt)}$$

where the factors $L(\psi_0, t)$ are the characteristic polynomials of the Frobenius on the isotypical components of $H^1(X)$ as an $\mathbb{F}_p$ representation. That is,

$$\sum_{h \in \mathbb{F}_{p^n}} \psi_0(\mathrm{Tr}_n(-ah^2)) = \mathrm{Tr}(F|(H^1(X))^{\psi_0}).$$

The zeta function of X can be computed using quadratic forms.

$$|X(\mathbb{F}_{p^n})| - 1 = p \cdot |\{\text{roots of } \mathrm{Tr}_n(-ah^2) \text{ in } \mathbb{F}_{p^n}\}|.$$

Fixing a $\mathbb{F}_p$ vector space basis for $\mathbb{F}_{p^n}$, we may consider $\mathrm{Tr}_n(-ah^2)$ as a quadratic form on $\mathbb{F}_p^n$. Standard results on representing zero state that

$$|X(\mathbb{F}_{p^n})| - 1 = p^n \pm (p-1)p^{n/2}$$

when n is even and

$$|X(\mathbb{F}_{p^n})| - 1 = p^n + \sqrt{p} \cdot p^{n/2}$$

when n is odd. The operation '$\pm$' depends on the determinant of the quadratic form one obtains and on a.

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Table 1. $U \cap {}^nU$

n	$U \cap {}^nU$
t	$u(h, f, d, e)$
$t \cdot s_1$	$u(h, f, 0, e)$
$t \cdot s_2$	$u(0, f, d, e)$
$t \cdot s_1 s_2$	$u(0, f, d, 0)$
$t \cdot s_2 s_1$	$u(h, 0, 0, e)$
$t \cdot s_1 s_2 s_1$	$u(h, 0, 0, 0)$
$t \cdot s_2 s_1 s_2$	$u(0, 0, d, 0)$
$t \cdot s_1 s_2 s_1 s_2$	$u(0, 0, 0, 0)$

Table 2. Green Functions of $\mathrm{GSp}(4, \mathbb{F}_q)$ and $\mathrm{Sp}(4, \mathbb{F}_q)$

	A_1	A_2	A_{31}	A_{32}	A_5
$Q^G_{T_o}$	$(q^2+1)(q+1)^2$	$(q+1)^2$	$3q+1$	$q+1$	1
$Q^G_{T_{s_1}}$	$-(q^4-1)$	$-q^2+1$	$q+1$	$-q+1$	1
$Q^G_{T_{s_2}}$	$-(q^4-1)$	q^2+1	$-q+1$	$q+1$	1
$Q^G_{T_{(s_1 s_2)^2}}$	$(q^2+1)(q-1)^2$	$(q-1)^2$	$-q+1$	$-3q+1$	1
$Q^G_{T_{s_2 s_1}}$	$(q^2-1)^2$	$-q^2+1$	$-q+1$	$q+1$	1

Table 3. Centralizers $C_G(t)$

t	$C_G(t)$	Order of $C_G(t)$
$t = (1,1,1)$ or $(-1,-1,1)$	G	$q^4(q^4-1)(q^2-1)(q-1)$
$t = (-1,1,1), (1,-1,1), (-i,i,-1)$, or $(i,-i,-1)$	C_1	$q^2(q^2-1)^2(q-1)$
$t = (a,a,1)$ with $a^2 \neq 1$, $(1,1,-1)$, or $(-1,-1,-1)$	C_2	$q(q^2-1)(q-1)^2$
$t = (a,1,1)$ or $(a,-1,1)$ with $a^2 \neq 1$	C_3	$q(q^2-1)(q-1)^2$
$t = (a,b,\pm 1), a \neq b, a^2 \neq \pm 1, b^2 \neq \pm 1$	T	$(q-1)^3$

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