Computations of Spaces of Paramodular Forms of General Level

Jeffery Breeding II, Cris Poor, and David S. Yuen

Abstract. This article gives upper bounds on the number of Fourier-Jacobi coefficients that determine a paramodular cusp form in degree two. The level $N$ of the paramodular group is completely general throughout. Additionally, spaces of Jacobi cusp forms are spanned by using the theory of theta blocks due to Gritsenko, Skoruppa and Zagier. We combine these two techniques to rigorously compute spaces of paramodular cusp forms and to verify the Paramodular Conjecture of Brumer and Kramer in many cases of low level. The proofs rely on a detailed description of the zero dimensional cusps for the subgroup of integral elements in each paramodular group.

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1. Introduction

A finite dimensional vector space of Siegel modular forms, $M_k(\Gamma)$, is determined by a finite set of Fourier coefficients. Specifying such determining sets without necessarily knowing the dimension of the space is a problem that dates back to Siegel. A related problem asks for the number of Fourier-Jacobi coefficients that determines $M_k(\Gamma)$. Eichler showed that the Fourier-Jacobi coefficients up to index $\frac{2 \sqrt{2}}{3 \pi} \mu_n^2 \frac{1}{k}$ determine the space $M_k(\text{Sp}_g(\mathbb{Z}))$; here, $\mu_n$ is Hermite’s constant. Generalizations and improvements were given in [24] and [27] but were still restricted to level one. Such results provide one avenue to rigorous computations. The Paramodular Conjecture [4] has recently focused interest on the paramodular groups $K(N)$ in degree two.

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Theorem 1.2 below presents an upper bound on the number of Fourier-Jacobi coefficients that determine $S_k(K(N)^*, \chi)$, the vector space of weight $k$ paramodular cusp forms of level $N$ that are eigenforms under all the paramodular Atkin-Lehner involutions with signs specified by a character $\chi$ trivial on $K(N)$. Determining sets of Fourier coefficients are also given. The case of prime level in Theorem 1.2 was first proven in [21].

Our application of Theorem 1.2 is an appealing strategy for computing individual spaces of paramodular cusp forms. The Fourier-Jacobi coefficients of paramodular forms are Jacobi forms and so Theorem 1.2 allows us to control spaces of paramodular forms by spanning a finite set of spaces of Jacobi forms, which is more tractable. These rigorous computations for low $kN$, are given in Tables 1, 2 and 3.

Weight 2 paramodular cusp forms occur in the Paramodular Conjecture and, for composite levels, our dimension results are new. Nontrivial weight 3 paramodular cusp forms provide canonical divisors on the moduli space of abelian surfaces. When the moduli space is rational or unirational, it follows that these rigorous computations for low $kN$, are given in Tables 1, 2 and 3.

Let $\phi_j N \xi_j N$. 

**Definition 1.1.** The Jacobsthal function $j(N)$ is defined to be the smallest positive integer $m$ such that every sequence of $m$ consecutive positive integers contains an integer coprime to $N$.

**Theorem 1.2.** Let $k, N \in \mathbb{N}$. Let $\chi : K(N)^* \to \{\pm 1\}$ be a character trivial on $K(N)$. Let $f \in S_k(K(N)^*, \chi)$ be a common eigenfunction of the paramodular Atkin-Lehner involutions and have Fourier-Jacobi expansion

\[
f = \sum_{j=1}^{\infty} \phi_j N \xi_j N.
\]

Let $N = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ be the prime factorization of $N$ and set $\tilde{N} = p_1 \cdots p_t$. Choose $\mu \in \mathbb{N}$ such that $2\mu + 1 \geq j(\tilde{N}/p_i)$ for all $i$. Let $\kappa$ be 1 when $N$ is prime, 2 when $N$ is a composite prime power and $1 + \mu + \mu^2$ otherwise. If $\phi_j N = 0$ for $j \leq \kappa$, then $f = 0$. When $N$ is a composite prime power the inequality may be taken strictly.
We state the double coset decomposition for the important case of prime powers $q^n$ in Theorem 1.3. For $q \in \mathbb{N}$ and $x, y, z, M \in \mathbb{Z}$, define the following matrix:

\[
W(q; x, y, M, z) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & Mq^x & 1 \\
Mq^x & zM^2q^y & 0 & 1
\end{pmatrix}.
\]

**Theorem 1.3.** Let $q \in \mathbb{N}$ be prime. Let $r \in \mathbb{Z}$ be nonnegative. We have the disjoint double coset decomposition

\[
\text{Sp}_2(\mathbb{Z}) = \bigcup_{\mu, \nu, \bar{M}, \bar{z}} (K(q^n) \cap \text{Sp}_2(\mathbb{Z})) W(q; r - \mu - \nu, r - \nu, M, z) P_{2,0}(\mathbb{Z}),
\]

where the indices range over $0 \leq \mu \leq r/2$, $\bar{M} \in (\mathbb{Z}/q^r\mathbb{Z})^\times / \{ \pm 1 \}$, $0 \leq \nu \leq r - 2\mu$, and $\bar{z} \in (\mathbb{Z}/q^r\mathbb{Z})^\times / (\text{squares})$ for $2\mu + 2\nu \leq r$ and $\bar{z} \in (\mathbb{Z}/q^r\mathbb{Z})^\times / (\text{squares})$ for $2\mu + 2\nu > r$.

We now explain how Theorem 1.2 and its companions in section 7 make rigorous a strategy for computing spaces of paramodular forms that had hitherto been heuristic. Denote by $S_k(K(N))^\epsilon$ the subspace of $S_k(K(N))$ with eigenvalue $\epsilon = \pm 1$ under the Fricke involution. For a paramodular cusp form $f \in S_k(K(N))^\epsilon$, let $f(\tau, z) = \sum_{n=1}^{\infty} \phi_{Nm}(\tau, z)e(Nm\omega)$ be the Fourier-Jacobi expansion, so that the $\phi_{Nm}$ are Jacobi cusp forms of weight $k$ and index $Nm$. Following a method of Aoki [1] for $N = 1$, it was pointed out in [21] that the involution condition

\[
(1) \quad \forall n, m, r \in \mathbb{Z}, \ c(m, r; \phi_{Nm}) = \epsilon c(m, -r; \phi_{Nm})
\]

is very strong. There it was asked whether any sequence $\phi_{Nm}$ of Jacobi forms that satisfies (1) has a convergent series $\sum_{n=1}^{\infty} \phi_{Nm}(\tau, z)e(Nm\omega)$ and is thus the Fourier-Jacobi expansion of some paramodular form. The cases of $N = 2, 3, 4$ were answered affirmatively in [21]. An improved upper bound on the dimensions of Atkin-Lehner subspaces $S_k(K(N))^\epsilon, \chi$ is obtained by combining the involution condition [1], and similar conditions for the other paramodular Atkin-Lehner involutions, with a determining set of Fourier-Jacobi coefficients. If the upper bounds are in fact the correct dimensions then $S_k(K(N))^\epsilon$ can be rigorously computed by the construction of enough cusp forms. Of course, any algorithm, however foolish, that computes upper bounds can make the same claim. The real point is that, in the examples where we can say for sure, our improved upper bound is in fact the correct dimension. The examples in this article are thus further evidence that involution conditions alone imply convergence. Theoretical work has yet to explain the success of this strategy; for interesting work on similar topics see [33] and [5]. The method for computing upper bounds of $\dim S_k(K(N))^\epsilon$ given here is appealing because it works for general levels $N$ and also for weight 2, a weight inaccessible to trace formulas in degree two.

In order to span spaces of Jacobi cusp forms $J^{\text{cusp}}_{k, N}$, we used the theory of theta blocks introduced by Gritsenko, Skoruppa and Zagier in [14]. Spaces of Jacobi cusp forms are not always spanned by cuspidal theta blocks but, following a suggestion of D. Zagier, we have had in our examples complete success spanning spaces of Jacobi forms by linear combinations of weak Jacobi forms that are theta blocks.

Our Table 1 of $\dim S_2(K(N))^\epsilon$ gives evidence for the truth of the Paramodular Conjecture of Brumer and Kramer [4]. Abelian surfaces $A$ defined over $\mathbb{Q}$ of conductor $N$ with $\text{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ should correspond to Hecke eigenforms $f \in S_2(K(N))^\epsilon$.
that are not Gritsenko lifts and that have rational eigenvalues. In this correspondence, we have $L(s, f, \text{spin}) = L(s, A, \text{Hasse-Weil})$. Furthermore, we should have $\epsilon = 1$ when the rank of $A$ is even and $\epsilon = -1$ when the rank of $A$ is odd. The smallest prime level for which $S_2(K(p))$ has a nonlift is $p = 277$, see [30]; the smallest known level for which $S_2(K(N))$ has a nonlift is $N = 249 = 3 \cdot 83$, see [32]. There are indeed abelian surfaces $A/\mathbb{Q}$ possessing these conductors and none for odd $N < 249$, see [4]. The rigorous computations given here provide more evidence for the Paramodular Conjecture in some cases $N \leq 60$ by showing that the low levels $N$ with no abelian surfaces over $\mathbb{Q}$ do not have rational paramodular nonlift eigenforms.

Table 1 gives $\dim S_2(K(N))$ for $N \leq 60$. In these tables, an omitted level $N$ indicates that the dimension is zero. Each table also gives the best a priori upper bound from section 7 on the needed number of Fourier-Jacobi coefficients. These weight two spaces were all spanned by Gritsenko lifts, which is consistent with the Paramodular Conjecture.

Table 1. Dimension of $S_2(K(N))$ and number of FJ-coefficients needed in proof.

<table>
<thead>
<tr>
<th>$N$</th>
<th>37</th>
<th>43</th>
<th>53</th>
<th>57</th>
<th>58</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>FJCs</td>
<td>7</td>
<td>8</td>
<td>10</td>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 2 gives the dimension of $S_3(K(N))$ for $N \leq 40$. These spaces were all spanned by Gritsenko lifts. All twenty levels with dimension zero are omitted.

Table 2. Dimension of $S_3(K(N))$ and number of FJ-coefficients needed in proof.

<table>
<thead>
<tr>
<th>$N$</th>
<th>13</th>
<th>17</th>
<th>19</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>25</th>
<th>26</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>FJCs</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 3 gives $\dim S_4(K(N))$ for $N \leq 40$ and the dimension of nonlifts.

Table 3. Dimension of lifts and nonlifts for $S_4(K(N))$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>7</th>
<th>10</th>
<th>11</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>FJCs</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>7</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
The signs in the tables label paramodular Atkin-Lehner spaces, see section 2 for the definition of the Atkin-Lehner involutions. Each row of signs is ordered by the distinct prime divisors of \( N \) and each sign in the row is the value of the Atkin-Lehner involution corresponding to the largest power of that prime dividing \( N \). For example, \( \dim \{ f \in S_4(K(34)) : f|AL_2 = -f \text{ and } f|AL_{17} = -f \} = 2 \). In every case covered by these tables, the product of the signs is \((-1)^k\), so the signs need not be listed when \( N \) is a power of a single prime.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( 22 )</th>
<th>( 23 )</th>
<th>( 24 )</th>
<th>( 25 )</th>
<th>( 26 )</th>
<th>( 27 )</th>
<th>( 28 )</th>
<th>( 29 )</th>
<th>( 30 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dim )</td>
<td>( 3^{1+2} )</td>
<td>( 3^{1+} )</td>
<td>( 3^{3+3} )</td>
<td>( 3^{3+2} )</td>
<td>( 4 )</td>
<td>( 2^{3+2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FJCs</td>
<td>( 4 )</td>
<td>( 8 )</td>
<td>( 5 )</td>
<td>( 12 )</td>
<td>( 7 )</td>
<td>( 11 )</td>
<td>( 5 )</td>
<td>( 10 )</td>
<td>( 3 )</td>
</tr>
</tbody>
</table>

| \( N \) | \( 31 \) | \( 32 \) | \( 33 \) | \( 34 \) | \( 35 \) | \( 36 \) | \( 37 \) | \( 38 \) | \( 39 \) | \( 40 \) |
|---|---|---|---|---|---|---|---|---|---|
| \( \dim \) | \( 6 \) | \( 3 \) | \( 4^{3+3} \) | \( 6^{3+3} \) | \( 4^{3+3} \) | \( 2^{3+2} \) | \( 8 \) | \( 5^{3+4} \) | \( 6^{3+4} \) | \( 5^{3+2} \) |
| FJCs | \( 12 \) | \( 14 \) | \( 7 \) | \( 10 \) | \( 4 \) | \( 15 \) | \( 14 \) | \( 8 \) | \( 12 \) | \( 10 \) |
| nonlift | \( 1 \) | \( 0 \) | \( 0 \) | \( 1^{++} \) | \( 0 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1^{++} \) | \( 1^{--} \) |

### 2. Siegel modular forms and notation

We set \( J = \left( \begin{smallmatrix} 0 & I_0 \end{smallmatrix} \right) \). The general symplectic group of degree \( n \) over a ring \( R \) is

\[
\text{GSp}_n(R) := \{ g \in \text{GL}_{2n}(R) : \exists \nu \in \text{GL}_1(R) : g^tJg = \nu J \}.
\]

The subgroup with \( \nu = 1 \) is \( \text{Sp}_n(R) \). We refer to the textbook \([15]\) for the general theory of Siegel modular forms. Let \( \Gamma \) be a discrete subgroup of \( \text{Sp}_n(\mathbb{R}) \) commensurable with \( \Gamma_n = \text{Sp}_n(\mathbb{Z}) \). Let \( \mathcal{H}_n \) be the Siegel upper half space and \( j : \text{Sp}_n(\mathbb{R}) \times \mathcal{H}_n \to \mathbb{C}^\times \) be the factor of automorphy given by \( j(g, Z) = \det(CZ+D) \) for \( g = (A \ B \ C \ D) \). We write \( M_k(\Gamma, \chi) \) for the \( \mathbb{C} \)-vector space of Siegel modular forms of weight \( k \) and character \( \chi \) with respect to \( \Gamma \); that is, the holomorphic functions on \( \mathcal{H}_n \), bounded at the cusps, that transform by the factor of automorphy \( j^k \chi \). The subspace of cusp forms is denoted by \( S_k(\Gamma, \chi) \).

In the case of degree one, we consider \( \Gamma_0(N) = (N^* \ 1) \cap \text{Sp}_1(\mathbb{Z}) \), for \( * \in \mathbb{Z} \), and the extensions \( \bar{\Gamma}_0(N) = \langle \Gamma_0(N), -I_2 \rangle \) and \( \Gamma_0(N) = \langle \Gamma_0(N), F_N \rangle \), where \( F_N = \frac{1}{\sqrt{N}} \left( \begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix} \right) \) is the Fricke involution. Additionally we consider the group \( \Gamma_1^0(N) \) obtained by adjoining all the Atkin-Lehner involutions. For \( \alpha \mid |N| \), meaning \( \alpha \mid N \) and \( (\alpha, \frac{N}{\alpha}) = 1 \), fix any \( x_1, x_2, t_1, t_2 \in \mathbb{Z} \) such that \( x_1x_2\alpha - t_1t_2 \frac{N}{\alpha} = 1 \) and set \( AL_\alpha = \frac{1}{\sqrt{\alpha}} \left( \begin{smallmatrix} x_1 & x_2 \\ t_1 \alpha & t_2 \end{smallmatrix} \right) \) and define \( \bar{\Gamma}_1^0(N) = \langle \bar{\Gamma}_0(N), \{ AL_\alpha \}_{\alpha \mid |N|} \rangle \).

For any \( N \in \mathbb{N} \), define the paramodular group in degree two to be

\[
K(N) = \left\{ \left( \begin{smallmatrix} * & * & * & * \\ * & * & N^* \ 1 \\ * & N^* \ N^* \ 1 \\ * & * \ 1 \ 1 \\ \end{smallmatrix} \right) : * \in \mathbb{Z} \right\} \cap \text{Sp}_2(\mathbb{Q})
\]
The group $K(N)$ has a normalizer given by $\mu_N = \left( \begin{array}{cc} F_N^* & 0 \\ 0 & F_N \end{array} \right)$ and we also call $\mu_N$ the Fricke involution. If we adjoin $\mu_N$, we get the group $K(N)^+ = \langle K(N), \mu_N \rangle$. For $\alpha || N$, $\mu_\alpha = \left( \begin{array}{cc} AL^\alpha & 0 \\ 0 & AL_\alpha \end{array} \right)$ is also a normalizer of $K(N)$ with $\mu_\alpha^2 \in K(N)$, see [10]. We let $K(N)^* = \langle K(N), \{\mu_\alpha\}_{\alpha || N} \rangle$ denote the extension of the paramodular group by all these paramodular Atkin-Lehner involutions. A character $\chi : K(N)^* \to \{\pm 1\}$ is called an Atkin-Lehner character if $\chi$ is trivial on $K(N)$. These characters form a group of order $2^t$, where $t$ is the number of distinct prime divisors of $N$. Throughout this article, when we write $M_k(K(N)^*, \chi)$, the weight $k$ is in $\mathbb{Z}_{\geq 0}$, the level $N$ is in $\mathbb{N}$ and the character $\chi$ is an Atkin-Lehner character.

We define, following [18], the standard groups $\Gamma_0^*(N) = K(N) \cap \text{Sp}_2(\mathbb{Z})$ and $P_1,0(R) = \{ (\begin{array}{cc} a & b \\ c & d \end{array}) : * \in R \} \cap \text{SL}_2(R)$, $P_2,1(R) = \{ (\begin{array}{cc} a & b \\ c & d \end{array}) : * \in R \} \cap \text{Sp}_2(R)$.

Define homomorphisms $i_1, i_2 : \text{SL}_2(\mathbb{R}) \to \text{Sp}_2(\mathbb{R})$ by

$i_1 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right)$, $i_2 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$.

and homomorphisms $u : \text{GL}_2(\mathbb{R}) \to \text{Sp}_2(\mathbb{R})$ and $t : M_{2 \times 2}(\mathbb{R}) \to \text{Sp}_2(\mathbb{R})$ by

$u(A) = \left( \begin{array}{cc} A & 0 \\ 0 & A^* \end{array} \right)$, $t(B) = \left( \begin{array}{cc} I & B \\ 0 & I \end{array} \right)$.

For $n \in \mathbb{N}$, define $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$, $\psi(n) = |\{x \in \mathbb{Z}/n\mathbb{Z} : x^2 = 1\}|$, $\varpi(n) = \phi(n)/(n\psi(n))$, $\xi(n) = \begin{cases} 1 & \text{if } n \leq 2, \\ 2 & \text{if } n \geq 3. \end{cases}$

3. Jacobi Forms

We define Jacobi forms following [9], the standard reference is [7]. Consider two types of elements in $\Gamma_2$, $h = u(\left( \begin{array}{cc} 1 & \lambda \\ \lambda & \gamma \end{array} \right))t(\left( \begin{array}{cc} 0 & \nu \\ 0 & 1 \end{array} \right))$ and $i_1 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ for $\lambda, \nu, \kappa \in \mathbb{Z}$, and for $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$. The subgroup of $\Gamma_2$ generated by the $h$ is called the Heisenberg group $H(\mathbb{Z})$. The character $\nu_H : H(\mathbb{Z}) \to \{\pm 1\}$ is defined by $\nu_H(h) = (-1)^{h+\lambda^2+\gamma^2+\nu^2}$. The second type gives a copy of $\text{SL}_2(\mathbb{Z})$ inside $\Gamma_2$. This copy of $\text{SL}_2(\mathbb{Z})$ along with $H(\mathbb{Z})$ and $\pm I_4$ generate $P_{2,1}(\mathbb{Z})$. The character $\nu_H$ extends uniquely to a character on $P_{2,1}(\mathbb{Z})$ that is trivial on the copy of $\text{SL}_2(\mathbb{Z})$. Likewise, the factor of automorphy $\epsilon(\sqrt{ct \tau} + d)$ of the Dedekind eta function extends uniquely to a factor of automorphy $P_{2,1}(\mathbb{Z}) \times (\mathcal{H}_1 \times \mathbb{C})$ that is trivial on $H(\mathbb{Z})$ and defines the multiplier $\epsilon : P_{2,1}(\mathbb{Z}) \to e(\frac{\tau}{2})$ where $e(x) = e^{2\pi i x}$. We select holomorphic branches of roots that are positive on the purely imaginary elements of $\mathcal{H}_n$.

For $m \in \mathbb{Q}$, $a, b, 2k \in \mathbb{Z}$, consider holomorphic $\psi : H_1 \times \mathbb{C} \to \mathbb{C}$ whose modified function $\tilde{\psi} : H_2 \to \mathbb{C}$, given by $\tilde{\psi}(\tau, z) = \psi(\tau, z)e(m\omega)$, transforms by the factor of automorphy $j^k e^{a/2} \nu_H$ for $P_{2,1}(\mathbb{Z})$. We necessarily have $2k \equiv a \equiv \mod 2$ and $m \geq 0$ for nontrivial $\psi$. Such $\psi$ have Fourier expansions $\psi(\tau, z) = \sum_{n, r \in \mathbb{Q}} c(n, r; \phi)q^{n}\zeta^r$, for $q = e^{i\pi}$ and $\zeta = e^{i\pi}$. The support of such $\psi$ is supp(\phi) = \{ (n, r) \in \mathbb{Q}^2 : c(n, r; \phi) \neq 0 \}. If the support of $\phi$ has $n$ bounded from below, we say $\phi$ is weakly holomorphic and write $\phi \in J_{\text{wh}}^*(\mathbb{C}^* \nu_H)$. Sometimes nearly holomorphic is used.
in place of \textit{weakly holomorphic} in the literature. We say \( \phi \) is \textit{weak} and write 
\( \phi \in J^\text{weak}_{k,m}(c^a v_H^b) \) if the support of \( \phi \) satisfies \( n \geq 0 \); \( \phi \in J_{k,m}(c^a v_H^b) \) if \( 4mn - r^2 \geq 0 \); \( \phi \in J^\text{mero}_{k,m}(c^a v_H^b) \) if \( 4mn - r^2 > 0 \). The notation \( J^\text{mero}_{k,m}(\chi) \) indicates the vector space of meromorphic functions on \( \mathcal{H} \times \mathbb{C} \) that transform like a Jacobi form of weight \( k \), index \( m \) and multiplier \( \chi \).

A generalized valuation due to [14] characterizes Jacobi forms from among weakly holomorphic Jacobi forms. Let \( \mathcal{G} = C^0(\mathbb{R}/\mathbb{Z})^{p,q} \) be the additive group of continuous functions \( g : \mathbb{R} \to \mathbb{R} \) that have period one and are piecewise quadratic. Define the positive (non-negative) elements in \( \mathcal{G} \) to be the semigroup of functions whose values are all positive (non-negative) in \( \mathbb{R} \); this makes \( \mathcal{G} \) a partially ordered abelian group. For \( \phi \in J^\text{wh}_{k,m}(\chi) \) and \( x \in \mathbb{R} \) define

\[
\ord(\phi; x) = \min_{(n, r) \in \text{supp}(\phi)} (n + rx + m x^2).
\]

The function \( \ord : J^\text{wh}_{k,m}(\chi) \to \mathcal{G} \), defined by \( \phi \mapsto \ord(\phi) \) is a generalized valuation in the sense that it satisfies

\[
\ord(\phi_1 \phi_2) = \ord(\phi_1) + \ord(\phi_2)
\]

on the ring of all weakly holomorphic Jacobi forms and

\[
\ord(\phi_1 + \phi_2) \geq \min(\ord(\phi_1), \ord(\phi_2))
\]

on each graded piece of fixed weight and index. See [14] for the following result:

\textbf{Theorem 3.1.} Let \( \phi \in J^\text{wh}_{k,m}(\chi) \). We have \( \phi \in J_{k,m}(\chi) \) if and only if the valuation is nonnegative, i.e., for all \( x \in [0, 1] \), \( \ord(\phi; x) \geq 0 \). We have \( \phi \in J^\text{mero}_{k,m}(\chi) \) if and only if the valuation is positive, i.e., for all \( x \in [0, 1] \), \( \ord(\phi; x) > 0 \). If \( m = 0 \), then \( \ord(\phi; x) \) is constant. If \( m > 0 \), then \( \ord(\phi; x) \) attains its minimum on \( 1/m \mathbb{Z} \).

The following dimension formulae are due to Skoruppa and Zagier [36].

\textbf{Theorem 3.3.} Let \( k, m \in \mathbb{N} \). Let \( \sigma_0(m) \) be the number of positive divisors of \( m \). Let \( \delta(k, m) \) be zero unless \( k = 2 \) and let \( \delta(2, m) = \frac{1}{2} \sigma_0(m) - 1 \) for non-square \( m \) and \( \delta(2, m) = \frac{1}{2} \sigma_0(m) - \frac{1}{2} \) for square \( m \). For even \( k \geq 2 \),

\[
\dim J^\text{mero}_{k,m} = \delta(k, m) + \sum_{j=0}^{m} \left( \dim S_{k+2j}(\text{SL}_2(\mathbb{Z})) - \frac{j^2}{4m} \right).
\]
For odd \( k \geq 3 \),

\[
\dim J_{k,m}^{\text{cusp}} = \sum_{j=1}^{m-1} \left( \dim S_{k+2j-1}(\text{SL}_2(\mathbb{Z})) - \left\lfloor \frac{j^2}{4m} \right\rfloor \right).
\]

## 4. Theta Blocks

The theory of theta blocks is due to Gritsenko, Skoruppa, and Zagier, see [14]. Theta blocks are useful for computing bases of spaces of Jacobi cusp forms.

**Definition 4.1** ([14]). A theta block is a function of the form

\[
\text{TB}(f) = \eta^{f(0)} \prod_{\ell \in \mathbb{N}} \left( \frac{\vartheta_{\ell}}{\eta} \right)^{f(\ell)}
\]

for a sequence \( f : \mathbb{N}_0 \to \mathbb{Z} \) of finite support, where \( \eta \) is the Dedekind eta function

\[
\eta(\tau) = \sum_{n \in \mathbb{N}} \left( \frac{12}{n} \right) q^{n^2/24}
\]

and we write \( \vartheta_{\ell}(\tau, z) = \vartheta(\tau, \ell z) \), where \( \vartheta \) is Jacobi’s odd theta function

\[
\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(2n^2+1)28 \zeta_{2n^2+1}^2}.
\]

**Proposition 4.2** ([14]). Let \( f : \mathbb{N}_0 \to \mathbb{Z} \) have finite support. Then

\[
\text{TB}(f) \in J_{k,m}^{\text{mero}}(\epsilon_{K, L})
\]

where

\[
2k = f(0), \quad L = \sum_{\ell \in \mathbb{N}} \ell f(\ell), \quad K = f(0) + 2 \sum_{\ell \in \mathbb{N}} f(\ell), \quad 2m = \sum_{\ell \in \mathbb{N}} \ell^2 f(\ell).
\]

To determine if \( \text{TB}(f) \) is a Jacobi cusp form, we have the formula

\[
\text{ord}(\text{TB}(f), x) = \frac{k}{12} + \frac{1}{2} \sum_{\ell \in \mathbb{N}} f(\ell) B_2(\ell x),
\]

proven in [14], where

\[
B_2(x) = (x - \lfloor x \rfloor)^2 - (x - \lfloor x \rfloor) + \frac{1}{6} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{(n\pi)^2}.
\]

We only consider theta blocks with trivial character, so that \( K \) is divisible by 24 and \( L \) is even. Moreover, we only consider theta blocks with no theta functions in the denominator, so that we here prefer the notation

\[
[p_1, \ldots, p_n]_k = \eta^{2k-n} \vartheta_{p_1}, \ldots, \vartheta_{p_n}.
\]

Due to these restrictions, \( \eta^{-6} \vartheta_{\ell_1} \vartheta_{\ell_2} \cdots \vartheta_{\ell_{10}} \) will be the form we use for weight two theta blocks, \( \eta^{-3} \vartheta_{\ell_1} \vartheta_{\ell_2} \cdots \vartheta_{\ell_{16}} \) for weight three, and \( \vartheta_{\ell_1} \vartheta_{\ell_2} \cdots \vartheta_{\ell_8} \) for weight four. Although theta blocks provide an efficient way to compute Jacobi cusp forms, cuspidal theta blocks do not always span the space. To complete a basis for the space, one might attempt to span other spaces of Jacobi forms and use the index raising and lowering operators. Following a suggestion of D. Zagier, the authors considered another method, which computes bases using only theta blocks of the same index.
There are more weak Jacobi forms than cusp forms and so we compute the theta blocks $\phi$ whose minimum order $\operatorname{Ord}(\phi)$ is “not too negative.” That is, for a fixed index, we find the theta blocks that have the largest negative minimum order. Sometimes a linear combination of theta blocks from this collection is a Jacobi cusp form. In the cases where we were unable to complete a basis in this manner, we moved onto the collection of theta blocks that had the second largest negative minimum order and tried to complete our basis by taking linear combinations of these theta blocks. This led to more spaces being spanned by linear combinations of theta blocks. If we still didn’t have a basis, then we moved on to the collection of theta blocks with the third largest negative minimum order. It turned out that the collection of theta blocks with the third largest negative minimum order was as far as we needed to go to span the spaces of Jacobi cusp forms for weights $k = 2, 3, 4$ up to the necessary indices. For this information, see the website [3].

Examples of our computations in the weight 2 case are provided below. In each example, we state the dimension of the space, theta blocks in some of the aforementioned collections, graphs of the valuation of the weak theta blocks on the interval $[0, 1]$, linear relations among the listed theta blocks, and a basis in terms of linear combinations of theta blocks. In the cases where we needed to use theta blocks that were weakly holomorphic, we list the terms with nonpositive determinant.

In order to search for weight 2 theta blocks, $[f_0, \ldots, f_{10}]$, first note that the index is $m = \frac{1}{2} \sum_{i=1}^{10} f_i^2$. Given the index $m$, we list all possible ways to write $2m$ as the sum of 10 squares and then graph the valuation $\operatorname{ord}(\psi(f); x)$. This is rigorous because Theorem 3.1 assures us that the minimum is attained on $\frac{1}{2m} \mathbb{Z}$. We discuss one example in detail because these graphs are informative.

Consider $j_5 = [1, 1, 2, 3, 4, 4, 5, 8, 14]_2 = \partial^2 \partial_2 \partial_3 \partial_4 \partial_5 \partial_8 \partial_9 / \eta^6$. The sum of the squares is 348 and so the index is $m = 174$ and $j_5 \in J_2^{\text{weak}}$. The weak form $j_5$ is not a Jacobi form and the graph of $y = \operatorname{ord}(j_5; x)$, shown below, dips below the $x$-axis. The location of these negative minima pinpoint terms in the Fourier expansion of $j_5$ whose determinant is negative. Recall

$$\operatorname{ord}(j_5; x) = \min(n + rx + mx^2) = \min_{(n, r) \in \text{supp}(j_5)} \left( \frac{4mn - r^2}{4m} + m(x + \frac{r}{2m})^2 \right).$$

The biggest dip occurs at $x = \pm \frac{112}{2m}$, corresponding to a value of $r = \pm 112$. Evaluating $\operatorname{ord}(j_5; \frac{112}{348}) = -\frac{16}{4m}$ tells us that a term is supported with $D = -16$. From the value of the discriminant $-16 = D = 4mn - r^2 = 4 \cdot 174n - 112^2$, we calculate that $n = 18$. Indeed, the Fourier expansion of $j_5$ contains the terms $q^{18} \zeta^{\pm 112}$. The fidelity of the graph is not high enough to see whether or not it dips below the $x$-axis at $x = \pm \frac{56}{348}$ (it does), but formula (2) gives $\operatorname{ord}(j_5; \frac{56}{348}) = \frac{4}{14} \frac{1}{4m}$ and there is a bad term $-q^{1} \zeta^{50}$ with $D = 4 \cdot 174 \cdot 5 - 50^2 = -1$. Another inequivalent bad term with $D = -1$ is given by $-q^{19} \zeta^{115}$. In fact, the big dip is actually composed of two separate parabolic arcs meeting at the $x$-axis.
$y = \text{ord}(j_5; x)$

bad terms: $-q^5 \zeta^{\pm 59}, q^{18} \zeta^{\pm 112}, -q^{19} \zeta^{\pm 115}$.

A. Brumer and D. Zagier suggested the Riemann theta relations as a likely source of identities among theta blocks. Consider the identity $(R_5)$ from page 20 of [23],

$$2 \prod_{j=1}^{4} \vartheta(\tau, z_j) - \prod_{j=1}^{4} \vartheta(\tau, w_j) = \prod_{j=1}^{4} \theta_{00}(\tau, w_j) - \prod_{j=1}^{4} \theta_{01}(\tau, w_j) - \prod_{j=1}^{4} \theta_{10}(\tau, w_j),$$

where the complex four-tuples $z = (z_1, z_2, z_3, z_4)$ and $w = (w_1, w_2, w_3, w_4)$ are related by $z = wA$ for the orthogonal matrix

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

**Proposition 4.3.** Let $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) \in \mathbb{N}^4$ have $\ell_1 + \ell_2 + \ell_3 + \ell_4$ even. Setting $p = (\ell_1, \ell_2, \ell_3, \ell_4)A \in \mathbb{Z}^4$ and $m = (\ell_1, \ell_2, \ell_3, -\ell_4)A \in \mathbb{Z}^4$, we have

$$\prod_{j=1}^{4} \vartheta(\tau, \ell_jz) + \prod_{j=1}^{4} \vartheta(\tau, m_jz) = \prod_{j=1}^{4} \vartheta(\tau, p_jz).$$

**Proof.** The Jacobi theta functions $\theta_{00}(\tau, z)$, $\theta_{01}(\tau, z)$ and $\theta_{10}(\tau, z)$ are even in $z$. Therefore setting $w$ equal to $(\ell_1z, \ell_2z, \ell_3z, \ell_4z)$ or to $(\ell_1z, \ell_2z, \ell_3z, -\ell_4z)$ gives the right hand side of equation (3) the same value. Therefore we have

$$2 \prod_{j=1}^{4} \vartheta(\tau, p_jz) - \prod_{j=1}^{4} \vartheta(\tau, \ell_jz) = 2 \prod_{j=1}^{4} \vartheta(\tau, m_jz) + \prod_{j=1}^{4} \vartheta(\tau, \ell_jz),$$

which is equivalent to the conclusion. $\square$
Example 4.4. (Index 67) \( \dim J_{2,67}^{\text{cusp}} = 2 \). There are precisely three cuspidal theta blocks, namely
\[
f_1 = [1, 1, 1, 2, 3, 4, 4, 5, 5, 6]_2, \quad f_2 = [1, 1, 2, 2, 3, 3, 4, 4, 5, 7]_2, \\
f_3 = [1, 1, 1, 2, 2, 3, 3, 4, 5, 8]_2.
\]
The graphs \( y = \ord(f_i; x) \) of their valuations on \([0, 1]\) show, by staying strictly above the \( x \)-axis, that the \( f_i \) are cusp forms. Each of these blocks is in the span of the other two by the linear relation
\[
f_1 - f_2 + f_3 = 0.
\]
The dimension formula in Theorem 3.3 makes the verification of such identities trivial and so a basis for \( J_{2,67}^{\text{cusp}} \) is \( \{f_1, f_2\} \). Alternatively, this identity is a consequence of Riemann’s theta relation. Six of the ten entries for each of the \( f_i \) are common and the remaining four-tuples satisfy the relation \([2, 3, 4, 7]_2 = [1, 2, 3, 8]_2 + [1, 4, 5, 6]_2\), which follows from Proposition 4.3 by taking \( \ell = (8, 3, 2, 1) \), for instance.

Example 4.5. (Index 191) \( \dim J_{2,191}^{\text{cusp}} = 2 \). There is precisely one cuspidal theta block, namely
\[
f_1 = [1, 2, 3, 3, 4, 5, 6, 7, 8, 13]_2.
\]
The graph \( y = \ord(f_1; x) \) of its valuation on \([0, 1]\) is the following.
In this case, the cuspidal theta blocks do not produce a basis. So we consider the theta blocks with the largest negative minimum order, which is \(-5/764\). The theta blocks with this order, graphs of their valuation on \([0,1]\), and the offending terms that prevent them from being cuspidal are below.

\[
\begin{align*}
g_1 &= [1, 1, 3, 4, 5, 6, 7, 8, 9, 10]_2, \\
g_2 &= [1, 3, 3, 4, 4, 5, 7, 7, 8, 12]_2, \\
g_3 &= [1, 2, 3, 3, 5, 6, 8, 9, 12]_2, \\
g_4 &= [1, 1, 2, 3, 3, 4, 5, 5, 6, 16]_2.
\end{align*}
\]

We have the following linear relations

\[-f_1 - g_1 + g_2 = 0, \quad -f_1 + g_3 + g_4 = 0.\]

A basis for \(J_{2,191}^{\text{cusp}}\) is \(\{f_1, g_1 - g_4\}\). The linear relations again follow from Riemann’s theta relations. The first reduces to \([3, 4, 7, 12]_2 = [2, 3, 6, 13]_2 + [1, 6, 9, 10]_2\) and the second to \([4, 7, 8, 13]_2 = [1, 4, 5, 16]_2 + [3, 8, 9, 12]_2\). These follow from Proposition 4.3 regardless which of the three occurring four-tuples is selected to be \(\ell\). The result that \(\{f_1, g_1 - g_4\}\) is a basis requires that \(g_1 - g_4\) be proven to be a cusp form. Since \(g_1 - g_4\) is not a theta block but a linear combination of theta blocks of index \(m = 191\), we have little choice but to compute the Fourier expansion of \(g_1 - g_4\) to order \(q^{\lfloor m/4 \rfloor}\) and check the positivity of the determinant for each term.

**Example 4.6.** (Index 174) \(\dim J_{2,174}^{\text{cusp}} = 1\). There are no cuspidal theta blocks. The largest negative minimum order is \(-1/174\). There is precisely one theta block with this order, namely

\[g_1 = [1, 2, 3, 4, 4, 5, 7, 8, 8, 10]_2.\]

Since we only have one such theta block, we move on to the collection of blocks with the second largest negative minimum order, which is \(-3/232\). The theta blocks
with this order are:

\[ h_1 = [1, 2, 3, 4, 5, 5, 7, 7, 7, 11]_2; \quad h_2 = [1, 1, 2, 3, 3, 4, 5, 7, 15]_2. \]

The linear combination \( g_1 - h_1 + h_2 \) that cancels the bad coefficients is zero, so we move on to the collection of blocks with the third largest negative minimum order, which is \(-2/87\). The theta blocks with this order, graphs of their valuations on \([0, 1]\), and the offending terms that prevent them from being cuspidal are below.

\[ j_1 = [2, 2, 2, 3, 5, 7, 8, 8, 10]_2; \quad j_2 = [1, 2, 2, 5, 6, 7, 8, 10]_2; \]
\[ j_3 = [1, 2, 4, 4, 5, 6, 7, 8, 11]_2; \quad j_4 = [1, 1, 1, 2, 3, 4, 5, 7, 11, 11]_2; \]
\[ j_5 = [1, 1, 2, 3, 4, 4, 5, 8, 14]_2. \]

The graph of ord\((j_5; x)\) has already been discussed in detail. We have the following linear relations amongst the theta blocks in these collections and a basis for this space is \( \{j_1 + j_5\} \).

\[ g_1 - h_1 + h_2 = 0; \quad [5, 7, 7, 11]_2 = [1, 3, 3, 15]_2 + [4, 8, 8, 10]_2, \]
\[ g_1 - j_1 + j_2 = 0; \quad [2, 2, 3, 5]_2 = [1, 1, 2, 6]_2 + [1, 3, 4, 4]_2, \]
\[ h_1 - j_3 + j_4 = 0; \quad [4, 4, 6, 8]_2 = [1, 1, 3, 11]_2 + [3, 5, 7, 7]_2, \]
\[ g_1 - j_3 + j_5 = 0; \quad [4, 6, 7, 11]_2 = [1, 3, 4, 14]_2 + [3, 7, 8, 10]_2. \]
5. Formal Fourier-Jacobi expansions

The Fourier expansion of a paramodular form $f \in M_k(K(N))$ is of the form

$$f(Z) = \sum_T a(T; f) e\left(\langle Z, T \rangle\right).$$

Here we set $(A, B) = \text{tr}(AB)$ for symmetric matrices $A, B$. The summation is over semidefinite $T$ with $2T$ even and $\langle T, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ divisible by $N$. When $f$ is a cusp form, then the summation is over $T$ that are also definite and we use the notation $\mathcal{X}_2(N) = \{ \left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right) : n, r, m \in \mathbb{Z} \}$ for the summation indices.

The group $\hat{\Gamma}(N)$ is the transpose of the group $\hat{\Gamma}_0(N)$. For $U \in \hat{\Gamma}(N)$, we have $\det(U)^k a(U' TU; f) = a(T; f)$ because $u(\hat{\Gamma}(N)) \subseteq K(N)$ and the action of $\hat{\Gamma}(N)$ naturally stabilizes $\mathcal{X}_2(N)$ via $T \mapsto T[U] = U' TU$. By writing elements of $\mathcal{H}_2$ as $Z = \left(\begin{smallmatrix} z \\ \omega \end{smallmatrix}\right)$ and by expanding the Fourier series in terms of $\xi = c(\omega)$, one has

$$f(Z) = \sum_{m=0}^{\infty} \phi_{Nm}(\tau, z) \xi^{Nm}, \quad c(n, r; \phi_{Nm}) = a\left(\left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right); f\right),$$

where each $\phi_{Nm}$ is a Jacobi form of weight $k$ and index $Nm$. If $f$ is a cusp form, then the $\phi_{Nm}$ are all Jacobi cusp forms.

We now explain how to implement our method for computing $S_k(K(N)^*, \chi)$. Suppose we know that the space $S_k(K(N)^*, \chi)$ is determined by its first $\ell$ Fourier-Jacobi coefficients. Section 7 will show how to calculate such $\ell$ without already knowing the dimension of $S_k(K(N)^*, \chi)$. Let $V = \bigoplus_{j=1}^{r_{\text{cusp}}} J_{k, N_j}$. The map $F_{\ell} : S_k(K(N)^*, \chi) \to V$ that sends a paramodular form to its first $\ell$ Fourier-Jacobi coefficients is injective and each paramodular Atkin-Lehner involution $\mu_\alpha$ provides equations satisfied by the image of $S_k(K(N)^*, \chi)$ under $F_{\ell}$.

**Proposition 5.1.** For $\alpha \parallel N$, fix $x, y \in \mathbb{Z}$ such that $\alpha y - \frac{N}{\alpha} x = 1$. Let $\epsilon_\alpha = \pm 1$. Assume that $f \in S_k(K(N))$ satisfies $f|\mu_\alpha = \epsilon_\alpha f$ and let $\sum_{j=-\infty}^{\infty} \phi_{jN}^{\ell/N}$ be the Fourier-Jacobi expansion of $f$. For all $n, r, m \in \mathbb{Z}$ we have

$$\epsilon_\alpha c(n, r; \phi_{Nm}) =
\begin{cases}
\epsilon_\alpha a\left(\left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right); f\right), & \text{if } \alpha \parallel N \\
\epsilon_\alpha a\left(\left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right); f\right), & \text{if } \alpha \parallel N
\end{cases}$$

**Proof.** Define $U$ by setting $U^* = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0 & x \\ N & 0 \end{smallmatrix}\right)$, the paramodular Atkin-Lehner involution for $\alpha$ is $\mu_\alpha = u(U)$. We calculate

$$\epsilon_\alpha c(n, r; \phi_{Nm}) = \epsilon_\alpha a\left(\left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right); f\right) = a(T; f|\mu_\alpha),$$

where we set $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right)$. Furthermore, using $\det(U) = 1$,

$$a(T; f|\mu_\alpha) = a(T; f|u(U)) = \det(U)^k a(U' TU^{-1}; f) = a(U^* TU^{-1}; f).$$

Multiplying the three matrices gives

$$U^* TU^{-1} = \left(\begin{smallmatrix}
\alpha n + x r + x^2 \frac{N}{\alpha} m & Nn + \frac{N}{\alpha} x^2 r + \alpha y^2 r + xy Nm \\
Nn + \alpha y^2 r + \frac{N}{\alpha} x^2 r + xy Nm & (N \alpha n + \alpha y^2 r + y^2 \alpha m) N
\end{smallmatrix}\right).$$

Therefore we have $a(U^* TU^{-1}; f) = c\left((\alpha n + x r + x^2 \frac{N}{\alpha} m), 2Nn + (\alpha y + \frac{N}{\alpha} x)r + 2xy Nm; \phi_{N(\frac{N}{\alpha} n + yr + y^2 \alpha m)}\right)$. \(\square\)
For a fixed Atkin-Lehner character $\chi: K(N)^* \to \{\pm 1\}$, denote by $V(\chi)$ the elements $(\phi_N, \phi_{2N}, \ldots, \phi_{MN})$ of $V$ that satisfy equation (4) for all $\alpha \mid N$ with $\epsilon_{\alpha} = \chi(\mu_{\alpha})$ and $a\alpha + x^2 Nm \leq \ell$ and $Nn + yr + y^2 \alpha m \leq \ell$. The image of $S_k(K(N)^*, \chi)$ under the injection $FJ_{\ell}$ is contained in $V(\chi)$ and $\dim S_k(K(N)^*, \chi) \leq \dim V(\chi)$.

In section 4 many spaces of Jacobi cusp forms were spanned by theta blocks. Theta blocks have integral Fourier coefficients and the equations defining $\sum$ of theta series and Borcherds products. Constructions in low weights are typically the more difficult aspect; modular forms can be constructed in a great variety of ways and this has always been part of the charm of the subject. In this context, the most obvious techniques for constructing cusp forms are Gritsenko lifts, traces of theta series and Borcherds products. Constructions in low weights are typically the more challenging. We give one nontrivial example.

The first nonlift for weight four occurs at level $N = 40$ but this is covered by Ibukiyama’s dimension formula because $N$ is prime. The first nonlift that occurs at a non-squarefree level is at $N = 40$ and we illustrate its construction. Our upper bound for $\dim S_4(K(40))$ is 5 and the space of lifts is only four dimensional. Let $\Xi \in J_{4,40}^{\text{usp}}$ be given by $\Xi = [1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5]_4$. Let $V_2: J_{k,m} \to J_{k,2m}$ be the index doubling Hecke operator from [2], page 41. Let $\phi \in J_{4,40}^{\text{usp}}$ be given by $\phi = [1, 1, 1, 2, 4, 4, 5]_4$. Let $\psi \in J_{4,40}^{\text{ph}}$ be defined by $\psi = (\Xi - \phi[V_2]) / \phi$. The weakly holomorphic Jacobi form $\psi$ has integral Fourier coefficients and, by the theorem on page 29 of [32], gives a holomorphic Borcherds product $\text{Borch}(\psi) \in S_4(K(40))$. The Fourier-Jacobi expansion shows that $\text{Borch}(\psi)$ is not a Gritsenko lift and thus that $\dim S_4(K(40)) = 5$. See [32] for such constructions.

6. Vanishing theorems

We prove vanishing theorems for paramodular cusp forms. For prime levels these results were first given in [30]. We review the results of [26] [28] [29]. Let $P_n(\mathbb{R})$ and $P_n^{\text{semi}}(\mathbb{R})$ be the spaces of positive definite and semidefinite $n$-by-$n$ symmetric real matrices, respectively.

**Definition 6.1.** A function $\phi: P_n^{\text{semi}}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ is called type one if

- For all $s \in P_n(\mathbb{R})$, $\phi(s) > 0$,
- for all $\lambda \in \mathbb{R}_{\geq 0}$ and $s \in P_n^{\text{semi}}(\mathbb{R})$, $\phi(\lambda s) = \lambda \phi(s)$,
- for all $s_1, s_2 \in P_n^{\text{semi}}(\mathbb{R})$, $\phi(s_1 + s_2) \geq \phi(s_1) + \phi(s_2)$.

Type one functions are continuous on $P_n(\mathbb{R})$ and respect the partial order on $P_n^{\text{semi}}(\mathbb{R})$. Basic examples are: For $s \in P_n^{\text{semi}}(\mathbb{R})$, define

- $m(s) = \inf_{u \in \mathbb{Z}_n \setminus \{0\}} u^tsu$, the Minimum function,
- $\text{tr}(s) = \inf_{u \in \text{Gl}_n(\mathbb{Z})} \text{tr}(usu)$, the reduced trace,
- $\delta(s) = \det(s)^{1/n}$, the reduced determinant,
\[ w(s) = \inf_{a \in \mathcal{P}_n(R)} \frac{(a \, s)}{\max(a, s)}, \] the dyadic trace.

For \( n = 2 \), the dyadic trace of a Legendre reduced \( s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{P}_2(R) \) is given by \( w(s) = a + c - |b| \), see [24]. Legendre reduced means \( 2|b| \leq a \leq c \). For \( f \in M_k(\Gamma) \), we set \( \text{supp}(f) = \{ T \in \mathcal{P}_n(Q) : a(T; f) \neq 0 \} \).

**Theorem 6.2.** (Vanishing Theorem for general subgroups.) Let \( \phi \) be type one. For all \( n \in \mathbb{N} \) there exists a \( c_n(\phi) \in \mathbb{R}_{>0} \) such that: For any subgroup \( \Gamma \subseteq \text{Sp}_n(\mathbb{Z}) \) with finite index \( I \) and coset decomposition \( \text{Sp}_n(\mathbb{Z}) = \bigcup_{i=1}^I \Gamma M_i \), we have

\[ \forall \ k \in \mathbb{N}, \forall \ f \in S_k(\Gamma), \quad \frac{1}{\ell} \sum_{i=1}^{\ell} \inf \phi(\text{supp}(f|M_i)) > c_n(\phi) k \implies f \equiv 0. \]

For \( n = 2 \), we may take \( c_n(\phi) = \inf \phi \left( \left[ \frac{1}{10} \left( \begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix} \right) \right] \right) \).

**Proof.** This is Theorem 2.5 from [29], except for the last comment, which is Corollary 5.8 from [26]. \( \square \)

We can apply this theorem to \( \Gamma = \Gamma'_0(N) \subseteq \text{Sp}_2(\mathbb{Z}) \) and use the double coset decomposition of section 8 to get a vanishing theorem for cusp forms in \( S_k(\Gamma'_0(N)) \), which also applies to paramodular forms because \( \Gamma'_0(N) \subseteq K(N) \). We begin with some lemmas. For any \( \alpha \in \mathbb{N} \) with \( \alpha | N \), denote

\[ w_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Even though the single cosets representatives of \( \Gamma'_0(N) \setminus \text{Sp}_2(\mathbb{Z}) \) are enumerated with some determination in section 8, we only require the following lemma here. Recall, \( \phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times| \), \( \psi(n) = |\{ x \in \mathbb{Z}/n\mathbb{Z} : x^2 = 1 \}| \) and \( \varpi(n) = \phi(n)/(n\psi(n)) \).

**Lemma 6.3.** Let \( \alpha, N \in \mathbb{N} \) with \( \alpha | N \) and set \( \gamma = (\alpha, \frac{N}{\alpha}) \). We have the disjoint single coset decomposition

\[ \Gamma'_0(N)w_\alpha P_{2,0}(\mathbb{Z}) = \bigcup_{i=1}^{\kappa_\alpha} \Gamma'_0(N)w_\alpha u_i \]

for some \( u_i \in P_{2,0}(\mathbb{Z}) \) and for

\[ \kappa_\alpha = \alpha^2 \varpi(\gamma) N \prod_{\text{prime } p | N} (1 + \frac{1}{p}). \]

As \( \alpha \) ranges over \( \alpha | N \), we have distinct double cosets \( \Gamma'_0(N)w_\alpha P_{2,0}(\mathbb{Z}) \).

**Proof.** The first assertion is Lemma 8.19 and the second is Lemma 8.20. \( \square \)

**Remark 6.4.** The significance of using double cosets where the right hand group is \( P_{2,0}(\mathbb{Z}) \) is that when \( u_i \in P_{2,0}(\mathbb{Z}) \), and when \( \phi \) is a class function, then

\[ \sum_{i=1}^{\kappa} \phi(\text{supp}(f|wu_i)) = \kappa \phi(\text{supp}(f|w)). \]

**Lemma 6.5.** For \( \alpha, N \in \mathbb{N} \) with \( \alpha | N \), we have

\[ w_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K(N) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]
**Lemma 6.6.** Let $f \in S_k(K(N))$ be a cusp form. Then
\[
\text{supp}(f|w_\alpha) = \frac{1}{\alpha} \left( \frac{\sqrt{\alpha}}{0} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) \text{supp}(f) \left( \frac{\sqrt{\alpha}}{0} \begin{pmatrix} 0 & 1 \end{pmatrix} \right).
\]

**Proof.** Since $f \in S_k(K(N))$, then by Lemma 6.5
\[
f|w_\alpha = f \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right),
\]
and the latter has support $\left( \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \right) \text{supp}(f) \left( \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \right)$ and the result follows. \(\square\)

Note that for $T \in \text{supp}(f)$, the matrix $\left( \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} \end{pmatrix} \right) T \left( \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} \end{pmatrix} \right)$ remains half-integral and of the same determinant. Here is our first vanishing theorem.

**Theorem 6.7** (Vanishing Theorem I for paramodular cusp forms of arbitrary level). Let $f \in S_k(K(N))$. Let $\phi$ be a type one $\text{GL}_2(\mathbb{Z})$-class function. If $f \neq 0$, then
\[
\sum_{\alpha|N} \sqrt{\alpha} \varpi((\alpha, \frac{N}{\alpha})) \inf \left( \left( \frac{\sqrt{\alpha}}{0} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) \text{supp}(f) \left( \frac{\sqrt{\alpha}}{0} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) \right)
\]
\[
\leq \phi \left( \frac{1}{36} \left( \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \right) \right) kN^2 \prod_{\text{prime } q|N} (1 + \frac{1}{q^2}).
\]

**Proof.** Set $\Gamma_2 = \bigcup_{i=1}^I \Gamma_0'(N)\mathcal{M}_i$ with $I = |\Gamma_0'(N)\backslash \Gamma_2| = N^3 \prod(1 + \frac{1}{p})(1 + \frac{1}{p^2})$ but use only the subset of representatives given by $\Gamma_2 \supseteq \bigcup_{\alpha|N} \Gamma_0'(N)w_\alpha P_{2,0}(\mathbb{Z}) = \bigcup_{\alpha|N} \bigcup_{i=1}^I \Gamma_0'(N)w_\alpha u_{\alpha,i}$, for $u_{\alpha,i} \in P_{2,0}(\mathbb{Z})$. These single cosets are distinct by Lemma 6.3. We plug these distinct single cosets into Theorem 6.2 to get
\[
\sum_{\alpha|N} \alpha^2 \varpi((\alpha, \frac{N}{\alpha})) N \prod_{\text{prime } q|N} (1 + \frac{1}{q^2}) \inf \phi \left( \text{supp}(f|w_\alpha) \right)
\]
\[
\leq \phi \left( \frac{1}{36} \left( \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \right) \right) kN^3 \prod_{\text{prime } q|N} (1 + \frac{1}{q})(1 + \frac{1}{q^2}).
\]
The result follows from Lemma 6.6 and some simplification. \(\square\)

We remark that the single cosets used in the proof of Theorem 6.7 are all from the paramodular identity zero-dimensional cusp $K(N)P_{2,0}(\mathbb{Q})$.

**Corollary 6.8.** Let $f \in S_k(K(N))$. Let $\delta$ be the reduced determinant function. If $f \neq 0$, then
\[
\inf \delta(\text{supp}(f)) \leq k \sqrt{\frac{2}{15}} \left( \sum_{\alpha|N} \alpha \varpi((\alpha, \frac{N}{\alpha})) \right)^{-1} N^2 \prod_{\text{prime } q|N} (1 + \frac{1}{q^2}).
\]

**Proof.** This follows from Theorem 6.7 using the reduced determinant function. \(\square\)

Some cosets can be represented by the paramodular Atkin-Lehner involutions $\mu_\alpha$ defined in section 2. Denote $\rho_\alpha = u \left( \frac{1}{\sqrt{\alpha}} I_2 \right)$ and note $a(T; f|\rho_\alpha) = \alpha^{-k} a(\alpha T; f)$ so that $\text{supp}(f|\rho_\alpha) = \frac{1}{\alpha} \text{supp}(f)$.

**Lemma 6.9.** Let $\alpha, N \in \mathbb{N}$ with $\alpha|N$ such that $(\alpha, N/\alpha) = 1$. Then
\[
w_\alpha \in K(N)\mu_\alpha \rho_\alpha P_{2,0}(\mathbb{Z}).
\]
Proof. Let \( h \in \mathbb{Z} \) be such that \( ha \equiv 1 \mod \frac{N}{\alpha} \). We can directly verify that
\[
w_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h \alpha - 1}{h} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mu_{\alpha} \rho_{\alpha} \begin{pmatrix} x_{2\alpha} & -t_1 N \alpha & 0 & t_1 (h \alpha - 1) \\ -t_2 & x_1 & 0 & 0 \\ t_3 & 0 & x_1 & 0 \\ 0 & x_2 & 0 & 0 \end{pmatrix}.
\]
\( \square \)

**Theorem 6.10** (Vanishing Theorem II for paramodular cusp forms of arbitrary level). Let \( f \in S_k(K(N)) \). Let \( \phi \) be a type one \( GL_2(\mathbb{Z}) \)-class function. If \( f \neq 0 \), then
\[
\sum_{\alpha | N : (\alpha, \frac{N}{\alpha}) = 1} \alpha \inf \phi(\text{supp}(f|\mu_\alpha)) + \sum_{\alpha | N : (\alpha, \frac{N}{\alpha}) \neq 1} \alpha \varpi((\alpha, \frac{N}{\alpha})) \inf \phi \left( \left( \sqrt{\frac{\varpi}{\alpha}} \ 0 \ \frac{0}{\alpha} \right) \text{supp}(f) \left( \sqrt{\frac{\varpi}{\alpha}} \ 0 \ \frac{0}{\alpha} \right) \right) \leq \phi \left( \frac{1}{\sqrt{3}} \left( \frac{3}{1} \right) \right) kN^2 \prod_{\text{prime } q | N} (1 + q^2).
\]

Proof. For each \( \alpha | N \) with \( (\alpha, N/\alpha) = 1 \), we have by Lemma 6.9 that \( f|w_\alpha = f|\mu_\alpha \rho_\alpha u \) for some \( u \in P_{2,0}(\mathbb{Z}) \). Since \( \phi \) is a class function, then \( \phi(\text{supp}(f|\mu_\alpha \rho_\alpha u)) = \phi(\text{supp}(f|\mu_\alpha \rho_\alpha)) \) Thus
\[
\phi \left( \frac{1}{\alpha} \left( \sqrt{\frac{\varpi}{\alpha}} \ 0 \ \frac{0}{\alpha} \right) \text{supp}(f) \left( \sqrt{\frac{\varpi}{\alpha}} \ 0 \ \frac{0}{\alpha} \right) \right) = \phi(\text{supp}(f|w_\alpha)) = \frac{1}{\alpha} \phi(\text{supp}(f|\mu_\alpha)).
\]
Plugging this into Theorem 6.7 proves the result. \( \square \)

In practice, we might either ignore the terms where \( (\alpha, \frac{N}{\alpha}) \neq 1 \) or replace those terms by some constant lower bound. A simple corollary is the following.

**Corollary 6.11.** Let \( \phi \) be a type one \( GL_2(\mathbb{Z}) \)-class function. If \( f \neq 0 \), then
\[
\sum_{\alpha | N : (\alpha, \frac{N}{\alpha}) = 1} \alpha \inf \phi(\text{supp}(f|\mu_\alpha)) \leq \phi \left( \frac{1}{\sqrt{3}} \left( \frac{3}{1} \right) \right) kN^2 \prod_{\text{prime } q | N} (1 + q^2).
\]

**Theorem 6.12** (Vanishing Theorem III for paramodular cusp eigenforms of arbitrary level). Let \( \phi \) be a type one \( GL_2(\mathbb{Z}) \)-class function. Let \( f \in S_k(K(N)) \) be an eigenform for the involution \( \mu_\alpha \) for every \( \alpha | N \) with \( (\alpha, \frac{N}{\alpha}) = 1 \). Then \( f \neq 0 \) implies
\[
\inf \phi(\text{supp}(f)) \leq \phi \left( \frac{1}{\sqrt{3}} \left( \frac{3}{1} \right) \right) kN \prod_{q^r \| N} q^r + q^{r-2}.
\]

Proof. This follows from Corollary 6.11 and by noting that
\[
\sum_{\alpha | N : (\alpha, \frac{N}{\alpha}) = 1} \alpha = \prod_{q^r \| N} (q^r + 1). \ \square
\]

7. Determining numbers of Fourier-Jacobi coefficients

The determining sets of Fourier coefficients that were worked out in the previous section can be used to give upper bounds for the number of Fourier-Jacobi coefficients that determine a paramodular cusp form. The most direct approach is to simply count the number of Fourier-Jacobi coefficients needed to cover the Fourier coefficient indices required by the determinant bound of Corollary 6.8 or the bound for general type one class functions and Atkin-Lehner eigenforms of Theorem 6.12. To this end we make the following definitions.
Definition 7.1. Let $N \in \mathbb{N}$. For $T \in X_2(N)$, define $m_N(T)$, $m_N^+(T)$ and $m_N^-(T)$ to be the minimum of $\frac{1}{N} \langle gTg', (0, 0) \rangle$ over $g \in \Gamma_0(N), \Gamma_0^+(N)$ and $\Gamma_0^-(N)$, respectively. For $\lambda \in \mathbb{R}^+$ and a type one function $\phi$, define $J_N(\phi, \lambda)$, $J_N^+(\phi, \lambda)$ and $J_N^-(\phi, \lambda)$ to be the maximum of $m_N(T)$, $m_N^+(T)$ and $m_N^-(T)$, respectively, over $T \in X_2(N)$ satisfying $\phi(T) \leq \lambda$.

We note that these minima exist and are constant on $\hat{\Gamma}_0(N)$-orbits of $X_2(N)$. We do not address the existence of the maxima except in specific cases and in principle allow $+\infty$ as a maximum.

Theorem 7.2. Let $f \in S_k(K(N))$ have a Fourier-Jacobi expansion given by $f = \sum_{j=1}^\infty \phi_N j^{\xi j}$. If $\phi_N = 0$ for $j \leq J_N(\delta, \delta_0)$ then $f = 0$, where

$$\delta_0 = \min \left\{ \frac{\sqrt{2} N^2 \prod_{q \mid N} (1 + \frac{l}{q^2})}{\sum_{\alpha \mid N} \alpha \varpi((\alpha, \frac{N}{\alpha}))} \right\}.$$

For $f \in S_k(K(N))^\pm$, we have $f = 0$ if $\phi_N = 0$ for $j \leq J_N(\delta, \delta_0)$. For Atkin-Lehner eigenforms $f \in S_k(K(N)^*, \chi)$, we have $f = 0$ if $\phi_N = 0$ for $j \leq J_N(\delta, \delta_0)$.

Proof. By Theorem 6.8 it suffices to prove that $\det(g)^k a(T; f) = 0$ for $T \in X_2(N)$ with $\delta(T) \leq \delta_0$. Since $\delta(T) \leq \delta_0$, we have $m_N(T) \leq J_N(\delta, \delta_0)$ and $g'Tg = \left( n \wedge \frac{r}{2} \wedge m \right)$ for $m = m_N(T)$ and some $g \in \hat{\Gamma}_0(N)$ and $n, r \in \mathbb{Z}$. Thus

$$\det(g)^k a(T; f) = \left( \left( n \wedge \frac{r}{2} \wedge m \right); f \right) = c(n, r; \phi_N m) = 0$$

by $m \leq J_N(\delta, \delta_0)$ and the hypothesis. The other two cases are similar.

Theorem 7.3. Let $\phi$ be a type one function that is a $GL_2(\mathbb{Z})$-class function. Let $f \in S_k(K(N)^*, \chi)$ have Fourier-Jacobi expansion $f = \sum_{j=1}^\infty \phi_N j^{\xi j}$. If $\phi_N = 0$ for $j \leq J_N^*(\phi, \lambda)$ then $f = 0$, where

$$\lambda = \phi(\frac{1}{30} \left( \frac{3}{1} \right)) k N \prod_{q \parallel N} \frac{q^r + q^{r-2}}{q^r + 1}.$$

Proof. The proof is the same as the proof of Theorem 7.2 except we use Theorem 6.12 in place of Corollary 6.8.

The above two theorems are satisfactory when circumstances permit running a computer program to tabulate $J_N(\delta, \delta_0)$ or $J_N^*(\phi, \lambda)$. We next formulate upper bounds of theoretical interest in terms of the Jacobsthal function defined in the Introduction. Indeed, the upper bounds given here are further motivation for studying the growth of the Jacobsthal function $j(N)$. H. Iwaniec proved that $j(N) \in O((\ln N)^2)$, see [22]. The Jacobsthal function is labeled A048669 in the Online Encyclopedia of Integer Sequences. We begin with the following lemma.

Lemma 7.4. Let $N = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where $p_i \neq p_j$ for $i \neq j$ and $\alpha_i \geq 1$, be the prime factorization of $N$. Set $N = p_1^{\mu_1} \cdots p_e^{\mu_e}$ and let $\mu \in \mathbb{N}$ satisfy the condition: $2\mu + \geq j(N/p_i)$ for all $i$. Then, for all $A, B, N \in \mathbb{N}$ with $(A, B) = 1$, we have $J(N, \mu) = 1$ or there exists a $y \in \mathbb{Z}$ with $|y| \leq \mu$ such that $(A + By, N) = 1$.

Proof. Suppose that $(B, N) \neq 1$ and, by rearranging the prime factors of $N$, suppose that $p_{\ell-r}, p_{\ell-r+1}, \ldots, p_r | B$ but $p_1, \ldots, p_{\ell-r-1} \not| B$, where $r \geq 0$. Consider $j$ satisfying $1 \leq j \leq \ell - r - 1$. If $p_j | (A + y_1 B)$ and $p_j | (A + y_2 B)$, then $p_j | (y_1 - y_2)B$,
which implies that $p_j|(y_1 - y_2)$ since $p_j \nmid B$ for $1 \leq j \leq \ell - r - 1$. Suppose by way of contradiction that there does not exist a $y$ with $|y| \leq \mu$ such that $(p_1 \cdots p_{\ell-r-1}, A + yB) = 1$. That is, suppose for all $|y| \leq \mu$ that there exists $p_j|(A + yB)$ for $1 \leq j \leq \ell - r - 1$. Since $(B, p_1 \cdots p_{\ell-r-1}) = 1$, there exists $z$ such that $A + zB$ is a multiple of $p_1 \cdots p_{\ell-r-1}$.

Consider the $2\mu + 1$ consecutive numbers

$$-\mu - z, -(\mu - 1) - z, \ldots, -z, \ldots, (\mu - 1) - z, \mu - z.$$

Since $2\mu + 1 \geq j \left(\frac{N}{p_1} \right) \geq j \left(\frac{N}{p_{\ell-r} \cdots p_{\ell-1}}\right) = j(p_1 \cdots p_{\ell-r-1})$, then at least one of these is relatively prime to $p_1 \cdots p_{\ell-r-1}$, call that $y - z$, where $|y| \leq \mu$. For each $1 \leq j \leq \ell - r - 1$, since $p_j|(A + zB)$ and $p_j \nmid (y - z)B$, then $p_j \nmid (A + zB + (y - z)B)$, which is $p_j \nmid (A + yB)$. Thus $(p_1 \cdots p_{\ell-r-1}, A + yB) = 1$. Thus in any case, there exists a $y$ with $|y| \leq \mu$ such that $(p_1 \cdots p_{\ell-r-1}, A + yB) = 1$. Fix this $y$. Now, for $\ell - r \leq j \leq \ell$, then $p_j|B$ implies $p_j \nmid A$, which implies $p_j \nmid (A + yB)$. Thus $p_j \nmid (A + yB)$ for all $1 \leq j \leq \ell$. That is, $(A + yB, N) = 1$. □

In order to transfer our estimates from Fourier coefficients to Fourier-Jacobi coefficients we bound $m^\pm_N$ from above by a type one $GL_2(\mathbb{Z})$-class function.

**Lemma 7.5.** Let $N \in \mathbb{N}$. Choose $\mu \in \mathbb{N}$ to satisfy the condition in Lemma 7.4. For $T \in \mathcal{X}_2(N)$, we have

$$m^+_N(T) \leq m(T), \quad \text{if } N \text{ is prime},$$

$$m^+_N(T) < \hat{\text{tr}}(T), \quad \text{if } N \text{ is a prime power},$$

$$m^+_N(T) \leq \frac{1}{2}(1 + \mu + \mu^2) \hat{\text{tr}}(T), \quad \text{in general}.$$  

**Proof.** Let $T \in \mathcal{X}_2(N)$. There exists a $\sigma = \begin{pmatrix} A & -B \\ * & * \end{pmatrix} \in GL_2(\mathbb{Z})$ such that $T[\sigma]$ is Legendre reduced. Thus we may assume that $T[\sigma] = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a \geq c \geq 2b \geq 0$, if we choose the reduction conditions in this manner. In this version, $c = m(T)$.

Consider the case where $N = p$ is prime. If $p|B$ then $\sigma \in \Gamma^0(N)$ and $m_N(T) = m_N \left( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) = c/N \leq c = m(T)$. If $(B, p) = 1$ then choose $\beta \in \mathbb{Z}$ satisfying $\beta B \equiv 1 \mod p$ and note that $\begin{pmatrix} A & -B \\ * & * \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & A\beta \end{pmatrix} \in \hat{\Gamma}^0(N)$. Therefore $T$ is in the same $\hat{\Gamma}^0(N)$-orbit as

$$T \left[ \begin{pmatrix} A & -B \\ * & * \end{pmatrix} \right] \left[ \begin{pmatrix} 0 & 1 \\ 1 & A\beta \end{pmatrix} \right] = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & A\beta \end{pmatrix} = \begin{pmatrix} c & * \\ * & * \end{pmatrix}.$$  

Noting $F_N^* \left( \begin{pmatrix} c & * \\ * & * \end{pmatrix} \right) \cdot F_{Nc}^{-1} = \left( \begin{pmatrix} * & * \\ * & Nc \end{pmatrix} \right)$, we have $m^+_N(T) \leq c = m(T)$.

Now consider the case where $N = p^r$ is a prime power, so that $(B, N) = 1$ or $(A, N) = 1$. If $(B, N) = 1$, we choose $\beta$ as before and get $m^+_N(T) \leq c < c + a = \hat{\text{tr}}(T)$. If $(A, N) = 1$, then for some choice of $y \in \mathbb{Z}$ we have $\begin{pmatrix} A & -B \\ * & * \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in$...
\( \hat{T}^0(N) \) and \( \hat{T} \) is in the same \( \hat{T}^0(N) \)-orbit as
\[
T \left[ \begin{pmatrix} A & -B \\ * & * \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right] = \left( \begin{array}{cc} a & b \\ b & c \end{array} \right) \left[ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right] = \left( \begin{array}{cc} a & * \\ * & * \end{array} \right).
\]
Therefore, \( m_N^+ (T) \leq a < \tilde{\text{tr}}(T) \).

In the case of general \( N \) we use our hypothesis on \( \mu \) and Lemma 7.4 to obtain \( (B,N) = 1 \) or the existence of \( a \in \mathbb{Z} \) with \( |a| \leq \mu \) and \( (A+By,N) = 1 \). If \( (B,N) = 1 \) we proceed as before and obtain \( m_N^+ (T) \leq c < \tilde{\text{tr}}(T) < \frac{1}{2}(1+\mu+\mu^2)\tilde{\text{tr}}(T) \), making use of \( \mu > 0 \). For the main case, suppose \( (A+By,N) = 1 \) so that \( \alpha(A+By) = 1+\ell N \) for some \( \alpha, \ell \in \mathbb{Z} \). Set \( S = \left( \begin{array}{ccc} 1 & \alpha B \\ -y & \alpha A - \ell N \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) and check that
\[
\sigma S = \left( \begin{array}{ccc} A & -B \\ * & * \end{array} \right) \left( \begin{array}{cc} 1 & y \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} A+By & \alpha B \ell N \\ * & * \end{array} \right) \in \hat{T}^0(N).
\]
Then \( T \) is in the same \( \hat{T}^0(N) \)-orbit as
\[
T[\sigma S] = \left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) \left[ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right] = \left( \begin{array}{cc} a-2by+cy^2 & * \\ * & * \end{array} \right),
\]
and therefore \( m_N^+ (T) \leq a-2by+cy^2 \). If \( y = 0 \), we have \( m_N^+ (T) \leq a-2by+cy^2 = a < a + c < \frac{1}{2}(1+\mu+\mu^2)\tilde{\text{tr}}(T) \). If \( y \neq 0 \), we argue
\[
a-2by+cy^2 &= \frac{a+c}{2} - 2by + \frac{a+c}{2}y^2 + \frac{a-c}{2}(1-y^2) \\
&\leq \frac{a+c}{2} + \frac{a+c}{2}|y| + \frac{a+c}{2}y^2 + 0 = \frac{1}{2}(a+c)(1+|y|+|y|^2) \\
&\leq \frac{1}{2}\tilde{\text{tr}}(T)(1+\mu+\mu^2).
\]

Theorem 7.6 proves Theorem 1.2 of the Introduction.

**Theorem 7.6.** Let \( f \in S_\kappa(K(N)^*, \chi) \) have Fourier-Jacobi expansion
\[
f = \sum_{j=1}^{\infty} \phi_j \xi \chi^j N.
\]
Let \( N = \prod_{i}^{\alpha_i} p_i^{\alpha_i} \) be the prime factorization of \( N \) with \( \alpha_i \in \mathbb{N} \) and distinct primes \( p_i \) and set \( \hat{N} = \prod_{i}^{\alpha_i} p_i \). Choose \( \mu \in \mathbb{N} \) such that \( 2\mu + 1 \geq j(\hat{N}/p_i) \) for all \( i \). Let \( \kappa \) be 1 when \( N \) is prime, 2 when \( N \) is a composite prime power and \( 1 + \mu + \mu^2 \) otherwise. If \( \phi_j N = 0 \) for \( j \leq \kappa \frac{k}{10} \prod_{p^r || N} \frac{p^r + p^r - 2}{p^r + 1} \), then \( f = 0 \). When \( N \) is a composite prime power this inequality may be taken strictly.

**Proof.** As in Theorem 7.2 this follows from the Fourier coefficient bound for the reduced trace \( \tilde{\text{tr}} \) and from the bound \( m_N^+(T) \leq \frac{1}{2}\kappa \tilde{\text{tr}}(T) \) from Lemma 7.5. \( \square \)
Table 5. Examples of bounds on the number of Fourier-Jacobi coefficients.

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<th>(\mathcal{J}_N^-(\delta, \delta_0))</th>
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Theorem 7.6 should be viewed as a worst case scenario, the most theoretical among a similar group of theorems all proven by reduction to a determining set of Fourier coefficients. If we use Iwaniec’s bound on the Jacobsthal function, the required number of Fourier-Jacobi coefficients in Theorem 7.6 is \(O(\sqrt{N} \ln(N))^4\). Various bounds on a determining number of Fourier-Jacobi coefficients are given in Table 5. The column headed \(J_N(\delta, \delta_0)\) gives a number of Fourier-Jacobi coefficients sufficient to determine \(S_2(K(N))\); the column \(\mathcal{J}_N^+(\delta, \delta_0)\) determines \(S_2(K(N))^\pm\); and the columns: Jstl, \(\mathcal{J}_N^-(\delta, \delta_0)\), \(\mathcal{J}_N^+(\text{tr}, \lambda)\) and \(\mathcal{J}_N^-(w, \lambda)\) determine \(S_2(K(N)^* , \chi)\). In this last case, there are examples of each of the three type one functions \(\delta\), tr and \(w\) winning over the other two. A more extensive list can be seen at [3].

8. Coset representatives of \(\Gamma_0' (N)/\Sp_2(\mathbb{Z})/\P_{2,0} (\mathbb{Z})\)

This section provides detailed coset representatives for \(\Gamma_0' (N)/\Sp_2(\mathbb{Z})/\P_{2,0} (\mathbb{Z})\) and for \(\Gamma_0' (N)/\Sp_2(\mathbb{Z})\). We briefly mention Satake compactifications and cusps as they pertain to this article, compare [2][33]. For \(\Gamma \subseteq \Sp_2(\mathbb{Q})\) commensurable with \(\Sp_2(\mathbb{Z})\), let \(S(\Gamma/\H_2)\) be the Satake compactification of \(\Gamma/\H_2\). The one-dimensional cusps of \(S(\Gamma/\H_2)\) correspond bijectively with the double cosets of \(\Sp_2(\mathbb{Q})\) and the zero-dimensional cusps of \(S(\Gamma/\H_2)\) correspond bijectively with the double cosets of \(\Sp_2(\mathbb{Q})/\P_{2,0} (\mathbb{Q})\). In [34], Reefschläger classified the double cosets of \(\Sp_2(\mathbb{Q})/\P_{2,1} (\mathbb{Q})\) and the following is an easy corollary.

**Theorem 8.1.** (Reefschläger) Let \(N \in \mathbb{N}\). We have a disjoint union

\[
\Sp_2(\mathbb{Q}) = \bigcup_{m \in \mathbb{N} : m \mid N} K(N) \begin{pmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P_{2,1} (\mathbb{Q}),
\]

In particular, the number of 1-dimensional cusps is \(\tau(N)\).

The zero-dimensional cusps of \(K(N)\) were enumerated in [31] by finding all the zero-dimensional cusps of each one-dimensional cusp of \(K(N)\).

**Theorem 8.2** (Theorem 1.3, [31]). Let \(N, f, N_0 \in \mathbb{N}\) with \(N = f^2 N_0\) for \(N_0\) squarefree. We have a disjoint double coset decomposition

\[
\Sp_2(\mathbb{Q}) = \bigcup_{c,M} K(N) C_0 (Mc) P_{2,0} (\mathbb{Q}), \text{ with } C_0 (x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\((c, M) \in \prod_{c \in \mathbb{N}: c \mid f} (\mathbb{Z}/c\mathbb{Z})^* /\{\pm 1\}\).
where \( M \) is prime to \( N \) and gives the class of \( \overline{M} \) in \((\mathbb{Z}/c\mathbb{Z})^\times/\{\pm 1\}\). The number of 0-cusps is \( 1 + \lfloor f/2 \rfloor \).

It is known, see [12], that the index of \( \Gamma_0'(N) \) in \( \text{Sp}_2(\mathbb{Z}) \) is

\[
|\Gamma_0'(N)\setminus\text{Sp}_2(\mathbb{Z})| = N^3 \prod_{q|N} (1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}).
\]

We indicate the proof. Recall that \( \mathbb{P}^3(\mathbb{Z}/N\mathbb{Z}) \) consists of equivalence classes of relatively prime 4-tuples of integers, where \( [\alpha_i] \sim [\beta_i] \) means that there is an integer \( m \) prime to \( N \) with \( m\alpha_i \equiv \beta_i \mod N \) for all \( i = 1, 2, 3, 4 \). Sending the bottom row of an integral symplectic matrix to its projective class in \( \mathbb{P}^3(\mathbb{Z}/N\mathbb{Z}) \) identifies the cosets in \( \Gamma_0'(N)\setminus\text{Sp}_2(\mathbb{Z}) \) with this finite projective space. To see that this map is injective, we note the following lemma, which will be used again.

**Lemma 8.3.** Let \( \sigma \in \text{Sp}_2(\mathbb{Z}) \). Then \( \sigma \in \Gamma_0'(N) \) if and only if \( \sigma_{12}, \sigma_{32}, \sigma_{42} \) are all multiples of \( N \) or \( \sigma_{41}, \sigma_{42}, \sigma_{43} \) are all multiples of \( N \).

**Proof.** Assume that \( \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_2(\mathbb{Z}) \) satisfies \( A_{12}, C_{12}, C_{22} \equiv 0 \mod N \). We need to show that \( C_{21}, D_{21} \equiv 0 \mod N \). The symplectic condition that \( A'C' \) is symmetric gives us \( A_{22}C_{21} \equiv 0 \) and \( A'D' - C'B = I_2 \) gives us \( A_{22}D_{21} \equiv 0 \). It now suffices to show that \( A_{22} \) is prime to \( N \) and this follows from our hypothesis and \( \det(\sigma) = 1 \). Using \( CD' \) symmetric and \( AD' - BC' = I_2 \) will prove the alternate version. \( \square \)

This bottom row map is surjective because any primitive element of \( \mathbb{Z}^4 \) may be a selected row of an integral symplectic matrix. To count the number of elements in \( \mathbb{P}^3(\mathbb{Z}/N\mathbb{Z}) \), first consider the case where \( N = q^r \). The number of vectors in \( (\mathbb{Z}/q^r\mathbb{Z})^4 \) not all divisible by \( q \) is \( N^4(1 - \frac{1}{q^r}) \). Dividing this by \( |(\mathbb{Z}/q^r\mathbb{Z})^\times| = q^r(1 - \frac{1}{q^r}) \) yields the answer \( N^3(1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}) \). The index for general \( N \) follows by the Chinese Remainder Theorem.

Simpler but also useful is the disjoint union

\[
K(N) = \bigcup_{\Gamma_0(N)(a b \ c d) \in \Gamma_0(N) \setminus \text{SL}_2(\mathbb{Z})} \Gamma_0'(N)I_2 \left( \begin{array}{cc} a & b/N \\ c & d \end{array} \right).
\]

This shows \( |\Gamma_0'(N)\setminus K(N)| = |\Gamma_0(N)\setminus \text{SL}_2(\mathbb{Z})| = |\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})| = N \prod_{p|N}(1 + \frac{1}{p}) \). We also indicate the proof. For \( \sigma \in K(N) \), we know that \( g = \gcd(N, \sigma_{22}) \) and \( \alpha = N\sigma_{24} \) are relatively prime because

\[
1 = \det(\sigma) = \det \left( \begin{array}{ccc} N & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \sigma_{22} & \alpha/N \\ \cdot & N & \cdot \\ N & \cdot & \cdot \\ N & N & \cdot \end{array} \right) = \det \left( \begin{array}{ccc} \cdot & N/g & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & g & \cdot \end{array} \right).
\]

In particular, for \( \sigma \in \Gamma_0'(N) \), \( \sigma_{22} \) is relatively prime to \( N \). This allows us to define a map \( \Gamma_0'(N)\setminus K(N) \to \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \) by sending \( \sigma \) to \( [\sigma_{22} + \ell N, N\sigma_{24}] \) for any \( \ell \in \mathbb{Z} \) making \( \sigma_{22} + \ell N \) and \( N\sigma_{24} \) relatively prime. Equation (5) now follows because the representatives are inequivalent.

Our strategy to create detailed coset representatives for the zero-dimensional cusps of \( \Gamma_0'(N) \) is to first find representatives associated to each zero-dimensional
cusp of $K(N)$. The benefit of this approach is that it counts how many cosets belong to each zero-dimensional cusp of $K(N)$.

First, we will consider $N = q^r$, a power of a prime. By Theorem 8.2

\begin{equation}
\text{Sp}_2(\mathbb{Z}) = \bigcup_{m, M} (\Gamma_0'(q^r)K(q^r)C_0(Mm)P_{2,0}(Q)) \cap \text{Sp}_2(\mathbb{Z}),
\end{equation}

where the disjoint union is over $m = q^\mu$ with $0 \leq \mu \leq r/2$ and over $M$ in $(\mathbb{Z}/m\mathbb{Z})^\times/\{\pm 1\}$ with $M$ prime to $q^\mu$. Now, even though the following union might not be disjoint, we have

\begin{equation}
\Gamma_0'(q^r)K(q^r)C_0(Mq^\mu)P_{2,0}(Q) = \bigcup_{A \in \mathcal{A}} \Gamma_0'(q^r)AC_0(Mq^\mu)P_{2,0}(Q),
\end{equation}

where $\mathcal{A}$ is a complete set of representatives for $\Gamma_0'(q^r) \backslash K(q^r)$. By equation (5), we can take the representatives to be

\begin{equation}
\mathcal{A} = \{i_2 \begin{pmatrix} q^j & t/q^r \\ t_2q^r & x_2 \end{pmatrix},
\end{equation}

where $0 \leq j \leq r$, and $(q^j, t) = 1$, and $t_2, x_2 \in \mathbb{Z}$ are such that $q^jx_2 - tt_2 = 1$. For each $A \in \mathcal{A}$, the strategy is to factor

\begin{equation}
AC_0(Mq^\mu) = \text{gcd},
\end{equation}

where $g \in \Gamma_0'(q^r)$, $w \in \text{Sp}_2(\mathbb{Z})$ and $d \in P_{2,0}(Q)$ and to then conclude

\begin{equation}
(\Gamma_0'(q^r)AC_0(Mq^\mu)P_{2,0}(Q)) \cap \text{Sp}_2(\mathbb{Z}) = \Gamma_0'(q^r)wP_{2,0}(\mathbb{Z}).
\end{equation}

The next theorem addresses the case where $A$ has $t$ prime to $q$.

**Theorem 8.4.** Let $q, r, M \in \mathbb{N}$ with $q$ prime and $M$ prime to $q$. Let $j, \mu, t \in \mathbb{Z}$ with $0 \leq \mu \leq r/2$ and $0 \leq j \leq r$. Let $(t, q) = 1$ and let $i_2 \begin{pmatrix} q^j & t/q^r \\ t_2q^r & x_2 \end{pmatrix} \in K(q^r)$, so that $q^jx_2 - tt_2 = 1$. We have

\begin{equation}
\left(\Gamma_0'(q^r)i_2 \begin{pmatrix} q^j & t/q^r \\ t_2q^r & x_2 \end{pmatrix}C_0(Mq^\mu)P_{2,0}(Q)\right) \cap \text{Sp}_2(\mathbb{Z}) = \Gamma_0'(q^r)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Mq^{-\mu} & 1 & 0 \\ Mq^{-\mu} & 0 & 0 & 1 \end{pmatrix}P_{2,0}(\mathbb{Z}).
\end{equation}

This coset representative is independent of $j$ and $t$ and only depends on $\mu$ and $r$.

**Proof.** Since $j + r - 2\mu \geq 0$ and since $M^2t$ is prime to $q$, there exist $z, z_2 \in \mathbb{Z}$ such that $z M^2t + z_2 q^{j+r-2\mu} = 1$. Note that $q^j | (tt_2 + 1)$. We have the divisibility
\( t(t_2 + M^2 z) = tt_2 + zM^2 t \equiv -1 + 1 \equiv 0 \mod q^j \). Then \( q^j | (t_2 + M^2 z) \), and so

\[
g = \begin{pmatrix}
  q^{j-r-2\mu} & 0 & z & 0 \\
  0 & 1 & 0 & 0 \\
  -M^2 t & 0 & z_2 & 0 \\
  0 & q^{r-j}(t_2 + M^2 z) & 0 & 1
\end{pmatrix} \in \Gamma'_0(q^r).
\]

It can be verified directly that

\[
i_2 \left( \begin{pmatrix}
  tq^\nu / q^r \\
  x_2
\end{pmatrix} \right) C_0(Mq^\mu) \]

equals

\[
g \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & Mq^{r-\mu} & 1 & 0 \\
  Mq^{r-\mu} & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  z_2 & -Mq^\mu z & -z & 0 \\
  0 & M^tq^{-r+\mu} & q^j & 0 \\
  0 & 0 & q^{j+r-2\mu} & -Mtq^{-\mu} \\
  0 & 0 & Mzq^{-\mu} & z_2q^{r-2\mu}
\end{pmatrix}
\]

and the result follows. \( \square \)

Now we address the case where \( A \) has \( q^\nu | t \) for \( \nu \geq 1 \) and so necessarily \( j = 0 \). We replace \( t \) by \( tq^\nu \) so that, in this modified notation, \( t \) is still prime to \( q \).

**Theorem 8.5.** Let \( q, r, M \in \mathbb{N} \) and \( t \in \mathbb{Z} \) with \( t \) and \( M \) prime to \( q \). Let \( \mu, \nu \in \mathbb{Z} \) satisfy \( 0 \leq \mu \leq r/2 \) and \( 1 \leq \nu \leq r \). Let \( i_2 \left( \begin{pmatrix}
  1 \\
  0
\end{pmatrix} \right) \in K(q^r) \). If \( \nu + 2\mu \leq r \), then

\[
\Gamma'_0(q^r) i_2 \left( \begin{pmatrix}
  1 \\
  0
\end{pmatrix} \right) C_0(Mq^\mu) P_{2,0}(\mathbb{Q}) \cap \operatorname{Sp}_2(\mathbb{Z})
\]

\[
= \Gamma'_0(q^r) \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & Mq^{r-\mu-\nu} & 1 & 0 \\
  Mq^{r-\mu-\nu} & zM^2q^{r-\nu} & 0 & 1
\end{pmatrix} P_{2,0}(\mathbb{Z})
\]

for some \( z \in \mathbb{Z} \) with \( (z, q) = 1 \). If \( \nu + 2\mu \geq r \), then

\[
\Gamma'_0(q^r) i_2 \left( \begin{pmatrix}
  1 \\
  0
\end{pmatrix} \right) C_0(Mq^\mu) P_{2,0}(\mathbb{Q}) \cap \operatorname{Sp}_2(\mathbb{Z})
\]

\[
= \Gamma'_0(q^r) \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & Mq^\mu & 1 & 0 \\
  Mq^\mu & zM^2q^{2\mu} & 0 & 1
\end{pmatrix} P_{2,0}(\mathbb{Z})
\]

for some \( z \in \mathbb{Z} \) with \( (z, q) = 1 \). In the second case, the representative is independent of \( \nu \), and so is identical to that of the first case when \( \nu = r - 2\mu \). Thus no new coset representatives are created by the second case.
Proof. First, we consider the case where $2\mu + \nu \leq r$. Since $M^2t$ is prime to $q^{r-2\mu-\nu}$, there are $z, z_2 \in \mathbb{Z}$ such that $zM^2t + z_2q^{r-2\mu-\nu} = 1$. Note

$$g = \begin{pmatrix} q^{r-2\mu-\nu} & 0 & z & 0 \\ 0 & 1 & 0 & 0 \\ -M^2t & 0 & z_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma'_0(q^r).$$

It can be verified directly that $g^{-1} i_2 \begin{pmatrix} 1 & tq^{\nu-r} \\ 0 & 1 \end{pmatrix} C_0(Mq^\mu) \text{ equals}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & z_2 & -Mq^\mu z & -z & 0 \\ 0 & 1 & 0 & 0 & Mq^{\mu+\nu-r}t & 1 & 0 & q^{-r+\nu}t \\ 0 & Mq^{r-\mu-\nu} & 1 & 0 & 0 & 0 & q^{r-2\mu-\nu} & -Mq^{-\mu}t \\ Mq^{r-\mu-\nu} & zM^2q^{r-\nu} & 0 & 1 & 0 & 0 & Mzq^{\mu} & z_2 \end{pmatrix}$$

and the result for the case $2\mu + \nu \leq r$ follows.

Next, we consider the case where $2\mu + \nu \geq r$. Let $z, z_2 \in \mathbb{Z}$ be selected so that the relation $zM^2tq^{2\mu+\nu-r} + z_2 = 1$ holds. Note

$$g = \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & 0 \\ -M^2tq^{2\mu+\nu-r} & 0 & z_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma'_0(q^r).$$

It can be verified directly that $i_2 \begin{pmatrix} 1 & tq^{\nu-r} \\ 0 & 1 \end{pmatrix} C_0(Mq^\mu) \text{ equals}$

$$g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Mq^{\mu} & 1 & 0 \\ Mq^{\mu} & zM^2q^{2\mu} & 0 & 1 \end{pmatrix} \begin{pmatrix} z_2 & -Mq^\mu z & -z & 0 \\ Mqtq^{\mu+\nu-r} & 1 & 0 & tq^{-r+\nu} \\ 0 & 0 & 1 & -Mqt^{\mu+\nu-r} \\ 0 & 0 & Mzq^\mu & z_2 \end{pmatrix}$$

and the result for the case $2\mu + \nu \geq r$ follows. \hfill \Box

We have an immediate corollary that will be useful in computations.

**Corollary 8.6.** We have $W(q; r - \nu - \mu, r - v, M, z) \in K(q^r)C_0(Mq^\mu)P_{2,0}(Q)$.

*Proof.* This follows from Theorems 8.4 and 8.5 in view of equation (7). \hfill \Box

We combine Theorems 8.4 and 8.5 into the following proposition.
Proposition 8.7. We have the union, not necessarily disjoint,

$$\text{Sp}_2(\mathbb{Z}) = \bigcup_{\tilde{M}, \mu, \nu, z} \Gamma'_0(q^r) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Mq^r & 0 & 1 \\ Mq^r & zM^2q^r & 0 & 1 \end{pmatrix} P_{2,0}(\mathbb{Z}),$$

for $0 \leq \mu \leq r/2$, $0 \leq \nu \leq r - 2u$, $M \in (\mathbb{Z}/q^s\mathbb{Z})^\times/\{\pm 1\}$ and $z \in \mathbb{Z}$ prime to $q^r$.

Proof. By equations [1] and [2], $\text{Sp}_2(\mathbb{Z})$ is the union of $\Gamma'_0(q^r)wP_{2,0}(\mathbb{Z})$ for $w$ running over the double coset representatives displayed in Theorems 8.4 and 8.5. From Theorem 8.4, we have already restricted $M$ to $(\mathbb{Z}/q^s\mathbb{Z})^\times/\{\pm 1\}$. Theorem 8.5 covers the case where $\nu = 0$, although we have slightly adjusted the coset representative to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Mq^r & 0 & 1 \\ Mq^r & zM^2q^r & 0 & 1 \end{pmatrix} \in \Gamma'_0(q^r) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Mq^r & 0 & 1 \\ Mq^r & 0 & 0 & 1 \end{pmatrix}$$

in this case. When $\nu \geq 1$, the first case $\nu + 2\mu \leq r$ of Theorem 8.5 gives the result exactly as stated. When $\nu \geq 1$, the second case of Theorem 8.5 corresponds to setting $\nu = r - 2\mu$ in this proposition as mentioned in Theorem 8.5. \qed

We now work to get single coset decompositions of the double cosets in Proposition 8.7 and to determine when the double cosets coincide. Here is a formal but useful lemma.

Lemma 8.8. Let $G$ be a group with subgroups $H, U, T$ such that $uTu^{-1} \subseteq T$ for all $u \in U$. Fix $w \in G$. Suppose there exists a subgroup $U_0 \subseteq U$ such that

1. For all $u \in U$, $t \in T$, $wutw^{-1} \in H$ implies $u \in U_0$.
2. For all $u \in U_0$, there exists a $t \in T$ such that $wutw^{-1} \in H$.

Let $T_0 = w^{-1}Hw \cap T$. Let $\{u_\alpha\}$ be a complete set of distinct coset representatives for $U_0 \setminus U$, and let $\{t_\beta\}$ be a complete set of distinct coset representatives for $T_0 \setminus T$. Then $UT$ is a subgroup of $G$ and

$$HwUT = \bigcup_{\alpha, \beta} Hwt_\beta u_\alpha$$

is a disjoint coset decomposition.

Proof. First, $UT = TU$ makes the containment $HwUT \supseteq \bigcup_{\alpha, \beta} Hwt_\beta u_\alpha$ obvious. On the other hand, let $y = hwut \in HwUT$ for any $h \in H$, $u \in U$, and $t \in T$. Then $wut^{-1}u^{-1} \in U_0$ for some coset representative $u_\alpha$. There exists $t' \in T$ such that $ww^{-1}t'=w^{-1}h' \in H$. Then $y = hwut = hh'w(t')^{-1}u_\alpha t = hh'w(t')^{-1}u_\alpha t' \in H$. But $(t')^{-1}u_\alpha t' \in \{t_\beta\}$ for some $t'' \in T_0$ and for some coset representative $t_\beta$. So $t'' = w^{-1}h''w$ for some $h'' \in H$. Then $y = hh'w(t')^{-1}u_\alpha t_\beta$.

Second, we prove these cosets are disjoint. Suppose $Hwt_\beta u_\alpha \cap Hwt_\beta u_{\alpha'} = \emptyset$. Then $wt_\beta u_{\alpha}^{-1}t_\beta^{-1}w^{-1} \in H$. Then $wu_\alpha^{-1}w^{-1}u^{-1}t_\beta u_{\alpha}^{-1}w^{-1} \in H$ implies
Let \( u=ut^\beta t_\beta^{-1}w^{-1} \in U_0 \) which forces \( u_\alpha = u_{\alpha_0} \). Then \( wt_\beta t_\beta^{-1}w^{-1} \in H \) implies \( t_\beta t_\beta^{-1} \in T_0 \) which implies \( t_\beta = t_{\beta_0} \).

We will always apply this lemma with the choices \( G = \text{Sp}_2(\mathbb{Z}), H = \Gamma_0(q^v) \),

\[
U = \{ u(\sigma) : \sigma \in \text{GL}_2(\mathbb{Z}) \}, \quad T = \{ t \left( \frac{e}{f} , \frac{g}{f} \right) : e, f, g \in \mathbb{Z} \}.
\]

Note that we have \( uTu^{-1} \subseteq T \) for all \( u \in U \). Thus we only need to check conditions (1) and (2) to apply Lemma 8.8 with these choices.

We will have need of the following groups. For \( \alpha, \beta \in \mathbb{N} \), define

\[
\tilde{\Gamma}_0^0(\alpha, \beta) = \{(a \ b \ c \ d) \in \text{GL}_2(\mathbb{Z}) : a|c, \beta|b, \gcd(a, \beta)|(a - (ad - bc)d) \},
\]

\[
\Gamma^0_0(\alpha, \beta) = \tilde{\Gamma}_0^0(\alpha, \beta) \cap \text{SL}_2(\mathbb{Z}).
\]

It is not hard to see that these are groups. Note \( \Gamma_0^0(1, 1) = \Gamma_0(1) \) and \( \Gamma_0^0(1, \beta) = \Gamma^0_0(\beta) \). Recalling the function \( \psi(n) = |\{ x \in \mathbb{Z}/n\mathbb{Z} : x^2 = 1 \}| \), we remark that

\[
\psi(q^x) = \begin{cases} 
2 & \text{if } q > 2, \\
1 & \text{if } q^x = 2, \\
2 & \text{if } q^x = 4, \\
4 & \text{if } q = 2 \text{ and } x > 2.
\end{cases}
\]

**Lemma 8.9.** Let \( \alpha, \beta \in \mathbb{N} \). We have

\[
| (\Gamma_0(\alpha) \cap \Gamma^0_0(\beta)) \backslash \text{SL}_2(\mathbb{Z}) | = | \Gamma_0(\alpha \beta) \backslash \text{SL}_2(\mathbb{Z}) | = \alpha \beta \prod_{p|\alpha \beta} (1 + 1/p).
\]

Let \( \gamma = \gcd(\alpha, \beta) \). Then

\[
| \Gamma^0_0(\alpha, \beta) \backslash \text{GL}_2(\mathbb{Z}) | = 2 \frac{\phi(\gamma)}{\psi(\gamma)} \alpha \beta \prod_{p|\alpha \beta} (1 + 1/p).
\]

In particular, when \( \gamma = 1 \), we have

\[
| \Gamma^0_0(\alpha, \beta) \backslash \text{GL}_2(\mathbb{Z}) | = 2 \alpha \beta \prod_{p|\alpha \beta} (1 + 1/p).
\]

**Proof.** First, we have

\[
| \Gamma_0(\alpha) \cap \Gamma^0_0(\beta) \backslash \text{SL}_2(\mathbb{Z}) | = | (\Gamma_0(\alpha) \cap \Gamma^0_0(\beta)) \backslash \Gamma^0_0(\beta) | \cdot | \Gamma^0_0(\beta) \backslash \text{SL}_2(\mathbb{Z}) |.
\]

Conjugating by \( \left( \frac{1}{0} \frac{9}{\beta} \right) \), we get

\[
| (\Gamma_0(\alpha) \cap \Gamma^0_0(\beta)) \backslash \Gamma^0_0(\beta) | = | \Gamma_0(\alpha \beta) \backslash \Gamma_0(\beta) |.
\]

Taking transpose-inverses, we get \( | \Gamma^0_0(\beta) \backslash \text{SL}_2(\mathbb{Z}) | = | \Gamma_0(\beta) \backslash \text{SL}_2(\mathbb{Z}) | \). The first claim then follows from

\[
| \Gamma_0(\alpha \beta) \backslash \Gamma_0(\beta) | \cdot | \Gamma_0(\beta) \backslash \text{SL}_2(\mathbb{Z}) | = | \Gamma_0(\alpha \beta) \backslash \text{SL}_2(\mathbb{Z}) |.
\]

The second claim amounts to \( | \Gamma^0_0(\alpha, \beta) \backslash (\Gamma_0(\alpha) \cap \Gamma^0_0(\beta)) | = \phi(\gamma)/\psi(\gamma) \). So it suffices to verify that the representatives for \( \Gamma^0_0(q^a, q^b) \backslash (\Gamma_0(q^a) \cap \Gamma^0_0(q^b)) \) are the \( \left( \frac{a}{b} \right) \in \Gamma_0(q^a) \cap \Gamma^0_0(q^b) \) that range over \( (a, d) \mod \gamma \) with distinct values of \( ad^{-1} \mod \gamma \). Since we must have \( ad \equiv 1 \mod \gamma \), we want distinct \( a^2 \mod \gamma \). Thus \( a \) ranges over \( (\mathbb{Z}/q^\gamma \mathbb{Z})^\times \) modulo the square roots of 1 and there are \( \phi(\gamma)/\psi(\gamma) \) of these. \( \square \)
Corollary 8.10. Let \(x, y \in \mathbb{N} \cup \{0\}\). We have \(\lvert \Gamma_0^0(q^2, q^n) \setminus \text{GL}_2(\mathbb{Z}) \rvert\)

\[
\begin{cases}
2, & \text{if } x = y = 0, \\
2q^{x+y}(1 + 1/q), & \text{if } \text{min} (x, y) = 0 \text{ but } x + y > 0, \\
2q^{x+y-\min(x,y)}(1 + 1/q) \frac{(1-1/q)}{\psi(q^{x+y-\min(x,y)})}, & \text{otherwise}.
\end{cases}
\]

We are now ready to prove our main result on single coset decompositions for prime power levels given a double coset from Proposition 8.7.

Theorem 8.11. Let \(q\) be prime and \(r, M \in \mathbb{N}\). Let \(z, \mu, \nu \in \mathbb{Z}\) for \(0 \leq \mu \leq r/2\) and \(0 \leq \nu \leq r-2\mu\). Assume that \((q, Mz) = 1\). Let \(w = W(q; r-\mu - \nu, r-\nu, M, z)\). Set \(\mathcal{T} = t \left( M_{2 \times 2}(\mathbb{Z}) \right) \) and \(\mathcal{U} = u \left( \text{GL}_2(\mathbb{Z}) \right) \). We have the disjoint union

\[
\Gamma_0^0(q^r) w P_{2,0}(\mathbb{Z}) = \bigcup_{u, t} \Gamma_0^0(q^r) wtu
\]

where \(u\) are coset representatives of \(U_0 \setminus \mathcal{U}\) and \(t\) are coset representatives of \(T_0 \setminus \mathcal{T}\), where \(T_0 = (w^{-1} \Gamma_0^0(q^r) w) \cap \mathcal{T}\) and where the subgroups \(U_0\) corresponding to each \(w\) are as follows.

1. If \(2\mu + 2\nu \leq r\) then \(\lvert \mathcal{T}_0 \setminus \mathcal{T} \rvert = q^{\mu + \nu}\) and

\[
U_0 = \begin{cases}
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) & \text{if } \nu = 0, \\
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) \frac{(1-1/q)}{\psi(q^{\nu})} & \text{if } \nu > 0.
\end{cases}
\]

The total number of single cosets in this case is

\[
\begin{cases}
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) & \text{if } \nu = 0, \\
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) \frac{(1-1/q)}{\psi(q^{\nu})} & \text{if } \nu > 0.
\end{cases}
\]

2. If \(2\mu + 2\nu > r\) then \(\lvert \mathcal{T}_0 \setminus \mathcal{T} \rvert = q^{3\mu + 3\nu - r}\) and \(\exists B \in \text{SL}_2(\mathbb{Z})\) such that

\[
U_0 = \begin{cases}
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) & \text{if } \nu = 0, \\
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) \frac{(1-1/q)}{\psi(q^{\nu})} & \text{if } \nu > 0.
\end{cases}
\]

The total number of single cosets in this case is

\[
\begin{cases}
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) & \text{if } r - 2\mu - \nu = 0, \\
u \left( \frac{q^\nu}{\psi(q^{\nu})} \right) (1 + 1/q) \frac{(1-1/q)}{\psi(q^{\nu-2\mu})} & \text{if } r - 2\mu - \nu > 0.
\end{cases}
\]
Proof: For each of the two cases, we will give a $\mathcal{U}_0$ and prove this $\mathcal{U}_0$ satisfies the hypothesis of Lemma 8.8 and calculate $|\mathcal{U}_0|$. For $a, b, c, d \in \mathbb{Z}$ with $\epsilon = ad - bc = \pm 1$, and for $e, f, g \in \mathbb{Z}$ define

$$
\begin{pmatrix}
  a & b & 0 & 0 \\
  c & d & 0 & 0 \\
  0 & 0 & cd & -ce \\
  0 & 0 & -eb & ea
\end{pmatrix} \in \mathcal{U}; 
\begin{pmatrix}
  1 & 0 & e & f \\
  0 & 1 & f & g \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0
\end{pmatrix} \in \mathcal{T}.
$$

Call (1, 2), (3, 2), (4, 2) the three key entries of a 4-by-4 matrix. By Proposition 8.3, a matrix in $\text{Sp}_2(\mathbb{Z})$ is in $\Gamma_0(q^r)$ if and only if its key entries are multiples of $q^r$.

**Case (1):** $2\mu + 2\nu \leq r$. Using $\mu + \nu \leq r/2$, we note that the three key entries of $wutw^{-1} \mod q^r$ are

$$
(1, 2) : b - aeMq^{-\mu - \nu} - bfMq^{-\mu - \nu} - afM^2q^{-\nu}z - bgM^2q^{-\nu}z \\
(3, 2) : (1 - \epsilon)dMq^{-\mu - \nu} + \epsilon cM^2q^{-\nu}z \\
(4, 2) : (1 + \epsilon)bMq^{-\mu - \nu} + (d - ae)M^2q^{-\nu}z.
$$

If we assume $wutw^{-1} \in \Gamma_0(q^r)$, then the (1, 2) entry forces $q^{-\mu - \nu}|b$. We really should say that the divisibility by $q^r$ of the (1, 2) entry forces $q^{-\mu - \nu}|b$, but we will continue to speak in an abbreviated manner. Since $r - \mu - \nu \geq 1$, then $a$ and $d$ must be relatively prime to $q$. Let $b = b'q^{-\mu - \nu}$. The above entries mod $q^r$ become

$$
(1, 2) : (b' - aeM - afM^2zq^\mu)q^{-\mu - \nu} \\
(3, 2) : ((1 - \epsilon)d + \epsilon cMzq^\mu)Mq^{-\mu - \nu} \\
(4, 2) : (d - ae)M^2q^{-\nu}z.
$$

The (3, 2) entry says $q^\mu|(1 - \epsilon)$. Let us first consider the subcase $q^\mu \geq 3$ where only $\epsilon = 1$ is possible. The (3, 2) entry then implies $q^\mu|c$. Finally the (4, 2) entry implies $q^\mu|(a - d)$. This implies $u \in u(\Gamma_0(0, q^r, q^{-\mu - \nu})])$.

Next, in the subcase $q^\mu \leq 2$, both $\epsilon = \pm 1$ are possible. If $\epsilon = 1$, then just as above, we must have $u \in u(\Gamma_0(0, q^r, q^{-\mu - \nu})])$. If $\epsilon = -1$, the (3, 2) and (4, 2) entries say $q^\mu|(2dq^{-\mu} - cMz)$ and $q^\mu|(a + d)$, so that $a, b, c, d$ define an element of the set $H_0 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) : ad - bc = -1 : q^{\mu - \nu}b, q^\nu|(a + d), q^\mu|(2dq^{-\mu} - cMz)\}$. Setting $\tilde{H} = H_0 \cup \Gamma_0(q^r, q^{-\mu - \nu})$, we have $u \in u(\tilde{H})$ regardless of the sign of $\epsilon$.

We note that $\tilde{H}$ is a group and that $\Gamma_0(q^r, q^{-\mu - \nu})$ has index 2 in $\tilde{H}$. To see this, it suffices to construct an involution $Q$ of determinant $-1$ that normalizes $\Gamma_0(q^r, q^{-\mu - \nu})$ and has $H_0 = QT_0(0, q^r, q^{-\mu - \nu})$. If we choose $c_0 \in \mathbb{Z}$ so that $Mzc_0 = -2q^{-\mu} \mod q^r$ then $Q = \begin{pmatrix} c_0 & 0 \\ 0 & c_0^{-1} \end{pmatrix} \in H_0$ works.

Summarizing these two subcases, we can define $\mathcal{U}_0$ as stated in case (1) of the theorem hypothesis. Then we have proven that $wutw^{-1} \in \Gamma_0(q^r)$ implies $u \in \mathcal{U}_0$. Corollary 8.10 shows $|\mathcal{U}_0|\mathcal{U}| = \frac{|\Gamma_0(q^r, q^{-\mu - \nu})|}{\text{GL}_2(\mathbb{Z})}$ as claimed. Let us show that both the hypotheses of Lemma 8.8 are satisfied for this $\mathcal{U}_0$. For any $u \in \mathcal{U}_0$, we will produce a $t \in \mathcal{T}$ such that $wutw^{-1} \in \Gamma_0(q^r)$. Given such a $u$, the key entries (3, 2) and (4, 2) of $wutw^{-1}$ are multiples of $q^r$. We just need to show there exists $e, f, g$ such that the the remaining key entry (1, 2) above is a multiple of $q^r$. Just take $e \in \mathbb{Z}$ such that $(b' - aeM - afM^2zq^\mu)$ is a multiple of $q^{\mu + \nu}$; this
is possible because \(aM\) is relatively prime to \(q\). Thus the hypotheses of Lemma \(8.8\) are satisfied. We compute \(T_0\) by noting that the key entries of \(wutw^{-1}\) mod \(q^r\) are:

\[
(1, 2) : -eMq^{-\mu - \nu} - fM^2q^{-\nu}z
\]

\[
(3, 2) : 0
\]

\[
(4, 2) : 0
\]

and so \(t \in T_0\) if and only if \(q^{\mu + \nu}|(eM + fM^2q^\nu)\). Thus \(|T_0\setminus T| = q^{\mu + \nu}.

**Case (2):** \(2\mu + 2\nu > r\). Using \(\mu + \nu > r/2\), the three key entries of \(wutw^{-1}\) mod \(q^r\) are

\[
(1, 2) : b - aeMq^{-\mu - \nu} - bfMq^{-\mu - \nu} - afM^2q^{-\nu}z - bgM^2q^{-\nu}z
\]

\[
(3, 2) : -(c + df)M^3q^{2r-2\nu} + (1 - \epsilon)dMq^{-\mu - \nu} - (cf + dg)M^3q^{2r-2\nu}z + \epsilon M^2q^{-\nu}z
\]

\[
(4, 2) : -(ae + bf)M^2q^{2r-2\mu - 2\nu} + (1 + \epsilon)bMq^{-\mu - \nu} - (ce + af + df + bg)M^3q^{2r-2\mu - 2\nu} + (d - ae)M^2zq^{\nu} - (cf + dg)M^4q^{2r-2\nu}z^2.
\]

Consider also the linear combination of \((4, 2) - Mq^\mu z(3, 2) - Mq^{-\mu - \nu}(1, 2):

\[
(8) \quad eMq^{-\mu - \nu}(b - aMq^\mu z + dMq^\nu z - cM^2q^{2\mu}z^2).
\]

If we assume \(wutw^{-1} \in \Gamma'_0(q^r)\), then the \((1, 2)\) entry forces \(q^{-\mu - \nu}|b\). If \(\mu > 0\), then \(r - \mu - \nu \geq 1\) so that \(a\) and \(d\) must be relatively prime to \(q\), and the \((3, 2)\) entry forces \(q^\mu|(1 - \epsilon)\). So for any \(\mu \geq 0\), we have \(q^\mu|(1 - \epsilon)\).

Let us consider the subcase \(q^\mu \geq 3\) first, so that only \(\epsilon = 1\) is possible. The \((3, 2)\) entry then implies \(q^{-2\mu - \nu}|c\). Finally expression (8) implies \(q^{-2\mu - \nu}|(a - d)\) and \(q^{\mu + \nu}|(b - aMq^\mu z + dMq^\nu z - cM^2q^{2\mu}z^2)\). If we define

\[
H = \{(a b c d) \in \text{SL}_2(\mathbb{Z}) : q^{-\mu - \nu}|b, q^{-2\mu - \nu}|c, q^{-2\mu - \nu}|(a - d), q^{\mu + \nu}|(b + (d - a)Mq^\mu z - cM^2q^{2\mu}z^2)\}
\]

then \(u \in u(H)\). The conditions defining \(H\) look strange but precisely say that \(H = B^{-1}\Gamma_0^0(q^{\mu + \nu}, q^{-2\mu - \nu})B\) for \(B = \begin{pmatrix} 0 & M^2q^\mu \\ 1 & -1 \end{pmatrix}\).

In the subcase \(q^\mu \leq 2\), both \(\epsilon = \pm 1\) are possible. If \(\epsilon = 1\), then just as above, we have \(u \in u(H)\). If \(\epsilon = -1\), then entry \((3, 2)\) implies \(q^{-2\mu - \nu}|(2dq^\mu - cMz)\) and \(q^{-2\mu - \nu}|(a + d)\). Expression (8) implies \(q^{\mu + \nu}|(b + (d - a)Mq^\mu z - cM^2q^{2\mu}z^2)\). Therefore we set

\[
H_0 = \{(a b c d) \in \text{GL}_2(\mathbb{Z}) : ad - bc = -1, q^{-\mu - \nu}|b, q^{-2\mu - \nu}|(2dq^\mu - cMz), q^{-2\mu - \nu}|(a + d), q^{\mu + \nu}|(b + (d - a)Mq^\mu z - cM^2q^{2\mu}z^2)\}.
\]

One can prove that \(\tilde{H} = H_0 \cup H\) is a group and that \(|H \setminus \tilde{H}| = 2\) by constructing an involution \(Q \in H_0\) that normalizes \(H\) and satisfies \(H_0 = QH\). Choosing \(b_0\) such that \(Mz^2b_0 \equiv 2q^{-\mu} \mod q^{-2\mu - \nu}\), we check that \(Q = B^{-1}\begin{pmatrix} 1 & b_0 \\ 0 & -1 \end{pmatrix}\) works.

Summarizing these two subcases, define

\[
U_0 = \begin{cases} u(H) & \text{if } q^\mu \geq 3, \\
u(\tilde{H}) & \text{if } q^\mu = 1, 2. \end{cases}
\]
Thus $wutw^{-1} \in \Gamma_0(q')$ implies $u \in U_0$. Also, $|\mathcal{U}_0 \setminus \mathcal{U}| = \frac{\xi(q')}{2}|H \backslash \text{GL}_2(\mathbb{Z})|$, and from $H = B^{-1}G^1_0(q^{\mu+\nu}, q^{-2\mu-\nu})B$, we note $|\mathcal{U}_0 \setminus \mathcal{U}| = \frac{\xi(q')}{2}|G_0^1(q^{\mu+\nu}, q^{-2\mu-\nu}) \setminus \text{GL}_2(\mathbb{Z})|$ is as claimed if we make use of Corollary 8.10.

Now, for any $u \in \mathcal{U}_0$, we will produce a $t \in T$ such that $wutw^{-1} \in \Gamma_0(q')$. Given such a $u$, the expression (3) is already a multiple of $q'$. Thus it suffices to show there exists $e, f, g$ such that the remaining key entries (1,2) and (3,2) above are multiples of $q'$. We set $g = 0$ and leave $e, f$ variable. Dividing entry (1,2) by $-q^{-\mu-\nu}$ and entry (3,2) by $-q^{2\mu+2\nu-r}$, the needed equations become

$$aMe + (bM + aM^2q^\mu)f \equiv 0 \mod q^{\mu+\nu}$$

$$cM^2e + (dM^2 + cM^3q^\mu)f \equiv (1 - e)dMq^{\mu+\nu-r} + \epsilon cM^2q^{2\mu+\nu-r} \mod q^{2\mu+2\nu-r}.$$  

The inhomogeneous term is integral because when $\epsilon = -1$ the divisibilities defining $H_0$ give $q^{-2\mu-\nu}|(2q^{-\nu}d - cMz)$, and when $\epsilon = 1$ the divisibilities defining $H$ give $q^{-2\mu-\nu}|c$. Viewing the above as linear equations in the variables $e, f, g$, the determinant is $M^3$. Since $M^3$ is relatively prime to $q$, the above equations have an integral solution in $e, f$. Thus the hypotheses of Lemma 8.8 are satisfied.

We compute $T_0$ by noting that the key entries of $wutw^{-1} \mod q'$ are:

$$\begin{align*}
(1, 2) & : q^{-\mu-\nu}(-eM - fM^2q^\mu z) \\
(3, 2) & : q^{2r-2\mu-2\nu}(-fM^2 - gM^3q^\mu z) \\
(4, 2) & : Mq^{2r-2\mu-2\nu}(-eM - fM^2q^\mu z) + Mzq^{2r-\mu-2\nu}(-fM^2 - gM^3q^\mu z).
\end{align*}$$

Since the (4,2) entry is a linear combination of the (1,2) and (3,2) entries, we only need to enumerate the solutions of $e, f, g \mod q'$ to

$$q^{-\mu-\nu}(-eM - fM^2q^\mu z) \equiv 0 \mod q',$$

$$q^{2r-2\mu-2\nu}(-fM^2 - gM^3q^\mu z) \equiv 0 \mod q'.$$

For any of the $q^r$ choices of $g$, from the second equation, there are $q^{2r-2\mu-2\nu}$ choices of $f$, and from the first equation, there are $q^{-\mu-\nu}$ choices of $e$. Thus there are $q^{4r-3\mu-3\nu}$ solutions in $e, f, g$ and thus $|T_0 \setminus T| = q^{3\mu+3\nu-r}$. This completes the proof in every case.

**Corollary 8.12.** Let $q$ be prime and $r, M_1, M_2, \mu_1, \mu_2, \nu_1, \nu_2, z_1, z_2 \in \mathbb{N}$ such that $(q, z_1) = (q, M_i) = 1$. Assume $0 \leq \mu_i \leq r/2$ and $0 \leq \nu_i \leq r - 2\mu_i$ for $i = 1, 2$. If $(M_1, \mu_1, \nu_1) \neq (M_2, \mu_2, \nu_2)$, for $M_i \in (\mathbb{Z}/q^\mu \mathbb{Z})^\times/\{\pm 1\}$, then the double cosets $\Gamma_0(q')W(q; r - \mu_1 - \nu_1, r - \nu_1, M_i, z_i)P_{2,0}(\mathbb{Z})$, for $i = 1, 2$, are disjoint.

**Proof.** If $(M_i, \mu_i)$ are different for $i = 1, 2$, then by Corollary 8.6 and Theorem 8.2 the two double cosets in question are disjoint. So consider the case where $(M_i, \mu_i) = (M, \mu)$ for both $i = 1, 2$, and where $\nu_1 \neq \nu_2$. From Theorem 8.11 the double cosets $\Gamma_0(q')W(q; r - \mu - \nu_1, r - \nu_1, M, z_1)P_{2,0}(\mathbb{Z})$ and $\Gamma_0(q')W(q; r - \mu - \nu_2, r - \nu_2, M, z_2)P_{2,0}(\mathbb{Z})$ decompose into different numbers of single cosets and hence must be disjoint. This is because in the formula for the number of single cosets, as $\nu$ increases by 1, the formula increases by $q^2$ in one factor and can at most decrease by 2 elsewhere in the formula. \qed
Lemma 8.13. Let \( r \in \mathbb{N} \). We have \(|\Gamma_0^r(q^r) \backslash \text{Sp}_2(\mathbb{Z})| = q^{3r}(1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}) = \)

\[
\sum_{\mu=0}^{[r/2]} \sum_{\nu=0}^{r-2\mu} q^{r+2\nu}(1 + \frac{1}{q})\phi(q^\mu) \begin{cases} 
1 & \text{if } \nu = 0 \text{ or } \nu = r-2\mu \\
(1 - \frac{1}{q}) & \text{if } 0 < \nu < r-2\mu 
\end{cases}.
\]

Proof: We rewrite the double sum as

\[
\sum_{\mu=0}^{[r/2]} \sum_{\nu=0}^{r-2\mu} q^{r+2\nu-1} \begin{cases} 
1 & \text{if } \mu = 0 \\
\frac{q^\nu(q-1)}{q} & \text{if } \mu \neq 0 \\
\frac{q^2-1}{q} & \text{if } 0 < \nu < r-2\mu 
\end{cases}.
\]

First, we have

\[
\sum_{\nu=0}^{r-2\mu} q^{2\nu} \begin{cases} 
q + 1 & \text{if } \nu = 0 \text{ or } \nu = r-2\mu \\
\frac{q^2-1}{q} & \text{if } 0 < \nu < r-2\mu 
\end{cases}.
\]

Then the original double sum is

\[
q^{r-1} + q^{3r-2} + q^{3r-1} + q^{3r} + \sum_{\mu=1}^{[r/2]} q^{r+\mu-1} \frac{q - 1}{q} \begin{cases} 
q + 1 & \text{if } r-2\mu = 0 \\
1 + q^{2r-4\mu} + q^{2r-4\mu-1} + q^{2r-4\mu+1} & \text{otherwise} 
\end{cases}
\]

\[
= q^{r-1} + q^{3r-2} + q^{3r-1} + q^{3r} + \sum_{\mu=1}^{[r/2]} q^{r-2} \begin{cases} 
(q^2 - 1)q^\mu & \text{if } r-2\mu = 0 \\
(q-1)q^\mu + (q^3 - 1)q^{2r-4\mu-1} & \text{otherwise} 
\end{cases}.
\]

In the case where \( r \) is even, this becomes

\[
q^{r-1} + q^{3r-2} + q^{3r-1} + q^{3r} + q^{r-2}(q^2 - 1)q^{r/2} + q^{r-2} \sum_{\mu=1}^{r/2-1} ((q-1)q^\mu + (q^3 - 1)q^{2r-3\mu-1}),
\]

which simplifies to

\[
q^{r-1} + q^{3r-2} + q^{3r-1} + q^{3r} + q^{r-2}(q^2 - 1)q^{r/2} + q^{r-2}(q^{r/2} - q + (q^{2r-4} - q^{r-1/2})q^3),
\]

which simplifies to \( q^{3r}(1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}) \).
In the case where \( r \) is odd, the original double sum becomes

\[
q^{-1} + q^{3r-2} + q^{3r-1} + q^{3r} + q^{-2} \sum_{\mu=1}^{r/2-1/2} \left( (q-1)q^\mu + (q^3 - 1)q^{2r-3\mu-1} \right),
\]

which is

\[
q^{-1} + q^{3r-2} + q^{3r-1} + q^{3r} + q^{-2}(q^{r/2-1/2} - q + (q^{2r-4} - q^{r/2-5/2})q^3),
\]

which also simplifies to \( q^{3r}(1 + \frac{1}{q} + \frac{1}{q^2}) \).

\[\square\]

**Theorem 8.14.** Let \( q \) be prime. Fix \( M, \mu, \nu \) with \( 0 \leq \mu \leq r/2 \) and \( 0 \leq \nu \leq r - 2\mu \). Assume \((q, z) = (q, M) = (q, z') = 1\). We have equality of the double cosets

\[
\Gamma_0'(q^r)W(q; r - \mu - \nu, r - \nu, M, z)P_{2,0}(\mathbb{Z}) = \Gamma_0'(q^r)W(q; r - \mu - \nu, r - \nu, M, z')P_{2,0}(\mathbb{Z})
\]

if and only if either

1. \( z'z \) is a square mod \( q^\nu \) in the case of \( 2\mu + 2\nu \leq r \), or
2. \( z'z \) is a square mod \( q^{-2\mu-\nu} \) in the case of \( 2\mu + 2\nu > r \).

To get all possible \( \Gamma_0'(q^r)W(q; r - \mu - \nu, r - \nu, M, z)P_{2,0}(\mathbb{Z}) \) by varying \( z \), we can obtain complete disjoint double cosets by running \( z \) over

1. \( (\mathbb{Z}/q^\nu\mathbb{Z})^\times/\) (squares) in the case of \( 2\mu + 2\nu \leq r \) (there are \( \psi(q^\nu) \) representatives), or
2. \( (\mathbb{Z}/q^{-2\mu-\nu}\mathbb{Z})^\times/\) (squares) in the case of \( 2\mu + 2\nu > r \) (there are \( \psi(q^{-2\mu-\nu}) \) representatives).

**Proof.** First, suppose the two double cosets are equal. Then there exist integers \( a, b, c, d, e, f, g \) with \( ad - be = 1 \) such that

\[
W(q; r - \mu - \nu, r - \nu, M, z)u(a b c d e f g)W(q; r - \mu - \nu, r - \nu, M, z')^{-1} \in \Gamma_0'(q^r)
\]

The following key entries of the above matrix product must be 0 mod \( q^r \):

1. \( 1, 2 : b - q^{-\mu-\nu}(aeM - bM - (af + bg)M^2q^\nu z') \)
2. \( 3, 2 : M^2q^{2r-2\mu-2\nu}(-ce - df - cfMz'q^\mu - dgMz'q^\nu) + M^2cz'q^{-\nu} \)
3. \( 4, 2 : M^3q^{-\nu}(az' - dz) + q^{2r-2\mu-2\nu}(2bMq^{-r+\mu+\nu} - dgM^2z'q^\nu) - aeM^2z'q^\nu - bgM^3z'q^\mu - ceM^3zq^\mu - dfM^3zq^\nu - cfM^4zz'q^{2\mu} \).

Note that the (1, 2) entry implies that \( q^{r-\mu-\nu}b \). Then the expression \( 2bMq^{-r+\mu+\nu} \) in the (4, 2) entry is an integer, so that the (4, 2) entry has the form

\[
M^2q^{-\nu}(az' - dz) + q^{2r-2\mu-2\nu} \text{ (an integer)}
\]

In the case where \( 2\mu + 2\nu \leq r \), we have \( 2r - 2\mu - 2\nu \geq r \) so that the (3, 2) entry forces \( q^\nu|c \) and the (4, 2) entry forces \( q^\nu| (az' - dz) \), which is \( az' \equiv dz \mod q^r \). Then \( ad - be = 1 \) implies \( ad \equiv 1 \mod q^r \), and so

\[
z' \equiv adz' \equiv (dz)^2 \mod q^r.
\]

So \( z' \) is a square mod \( q^\nu \). There are \( \psi(q^\nu) \) elements in \((\mathbb{Z}/q^\nu\mathbb{Z})^\times/\) (squares) and hence there are at least \( \psi(q^\nu) \) distinct double cosets in this case.

In the case where \( 2\mu + 2\nu > r \), we have \( 2r - 2\mu - 2\nu < r \) so that the (3, 2) entry forces \( q^{2r-2\mu-2\nu}|c \) which forces \( q^{r-2\mu-\nu}|c \). In a similar way, the (4, 2) entry forces \( q^{r-2\mu-\nu}|(az' - dz) \). A calculation similar to the other case shows that \( z' \) is a square mod \( q^{r-2\mu-\nu} \), and hence there are at least \( \psi(q^{r-2\mu-\nu}) \) distinct double cosets in this case.
We have now proven that equality of the two double cosets implies that \( z z' \) is a square modulo an appropriate power of \( q \). For convenience, denote
\[
\psi(\mu, \nu) = \begin{cases} 
\psi(q^r) & \text{if } 2\mu + 2\nu \leq r \\
\psi(q^{r-2\mu-\nu}) & \text{if } 2\mu + 2\nu > r
\end{cases}
\]
and denote by \( \hat{\psi}(\bar{M}, \mu, \nu) \) the number of distinct double cosets that we obtain from the \( \Gamma_0'(q^r)W(q; r - \mu - \nu, r - \nu, M, z)P_{2,0}(\mathbb{Z}) \), for fixed \( \mu, \nu, \bar{M} \), as \( z \) varies over all integers. We just proved \( \hat{\psi}(M, \mu, \nu) \geq \psi(\mu, \nu) \) and we would like to prove equality.

We use this new notation to count single and double cosets. By Corollary 8.12, the double coset count is
\[
|\Gamma_0'(q^r)\backslash \text{Sp}_2(\mathbb{Z})/P_{2,0}(\mathbb{Z})| = \sum_{\mu = 0}^{\lfloor r/2 \rfloor} \sum_{\nu = 0}^{r-2\mu} \sum_{\bar{M} \in (\mathbb{Z}/q^r\mathbb{Z})^\times/\{\pm 1\}} \hat{\psi}({\bar{M}, \mu, \nu}).
\]
By Theorem 8.11, each double coset \( \Gamma_0'(q^r)W(q, r - \mu - \nu, r - \nu, M, z)P_{2,0}(\mathbb{Z}) \) has a single \( \Gamma_0'(q^r) \) cosets. To see this unification of the two cases in Theorem 8.11, just note that \( \nu = 0 \) implies \( 2\mu + 2\nu \leq r \) (Case I) and that \( \nu > 0 \), \( \nu = r - 2\mu \) imply \( 2\mu + 2\nu > r \) (Case II) and that \( \psi(\mu, 0) = \psi(\mu, r - 2\mu) = 1 \). The number of single cosets in each double coset is evidently independent of \( z \) and thus the total number of single cosets is
\[
|\Gamma_0'(q^r)\backslash \text{Sp}_2(\mathbb{Z})| = \sum_{\mu = 0}^{\lfloor r/2 \rfloor} \sum_{\nu = 0}^{r-2\mu} \sum_{\bar{M} \in (\mathbb{Z}/q^r\mathbb{Z})^\times/\{\pm 1\}} \frac{\hat{\psi}({\bar{M}, \mu, \nu})}{\psi(\mu, \nu)} \xi(q^r)q^{r+2\nu} \left( 1 + \frac{1}{q} \right) \left\{ \begin{array}{ll} 1 & \text{if } \nu = 0 \text{ or } \nu = r - 2\mu \\ (1 - \frac{1}{q}) & \text{if } 0 < \nu < r - 2\mu \end{array} \right\}.
\]
On the other hand, by Lemma 8.13
\[
|\Gamma_0'(q^r)\backslash \text{Sp}_2(\mathbb{Z})| = \sum_{\mu = 0}^{\lfloor r/2 \rfloor} \sum_{\nu = 0}^{r-2\mu} \sum_{\bar{M} \in (\mathbb{Z}/q^r\mathbb{Z})^\times/\{\pm 1\}} \xi(q^r)q^{r+2\nu} \left( 1 + \frac{1}{q} \right) \left\{ \begin{array}{ll} 1 & \text{if } \nu = 0 \text{ or } \nu = r - 2\mu \\ (1 - \frac{1}{q}) & \text{if } 0 < \nu < r - 2\mu \end{array} \right\}
\]
because \( |(\mathbb{Z}/q^r\mathbb{Z})^\times/\{\pm 1\}| = \phi(q^r)/\xi(q^r) \). Comparing these two formulae and using \( \psi(\bar{M}, \mu, \nu) \geq \psi(\mu, \nu) \), we see that \( \psi(\bar{M}, \mu, \nu) = \psi(\mu, \nu) \) for each summation index.

The following corollary is Theorem 1.3 of the Introduction.

**Corollary 8.15.** We have the disjoint union of double cosets
\[
\text{Sp}_2(\mathbb{Z}) = \bigcup_{\mu, \nu, \bar{M}, z} \Gamma_0'(q^r)W(q; r - \mu - \nu, r - \nu, M, z)P_{2,0}(\mathbb{Z})
\]
where \( 0 \leq \mu \leq r/2 \), \( \bar{M} \in (\mathbb{Z}/q^\nu \mathbb{Z})^\times/\{\pm 1\} \), \( 0 \leq \nu \leq r - 2\mu \), and we select indices \( \bar{z} \in (\mathbb{Z}/q^\nu \mathbb{Z})^\times/\{(\pm 1)\} \) for \( 2\mu + 2\nu \leq r \) and \( \bar{z} \in (\mathbb{Z}/q^{r-2\nu} \mathbb{Z})^\times/\{(\pm 1)\} \) for \( 2\mu + 2\nu > r \). The number of double cosets is

\[
\sum_{\mu=0}^{\lfloor r/2 \rfloor} \sum_{\nu=0}^{r-2\mu} \phi(q^\mu) \xi(q^\nu) \psi(q^{\min(r,2\mu-\nu)}).
\]

Lemma 8.16. Let \( q, N_2 \in \mathbb{N} \) with \( q \) prime and \( q \nmid N_2 \). We can choose representatives \( \zeta_1, \ldots, \zeta_\Psi(q^r) \in \Gamma_0(N_2') \) that give a disjoint union

\[
\text{Sp}_2(\mathbb{Z}) = \bigcup_{i=1}^{\Psi(q^r)} \Gamma_0(q^r) \zeta_i.
\]

Proof. By Corollary 8.15 double coset representatives \( w \) of \( \Gamma_0(q^r) \backslash \text{Sp}_2(\mathbb{Z}) / P_{2,0}(\mathbb{Z}) \) may be chosen in the form \( w = W(q;r - \mu, r - \nu, M, z) \). We cleverly choose \( M \in \mathbb{N} \), fixing \( M \in (\mathbb{Z}/q^\nu \mathbb{Z})^\times/\{\pm 1\} \), to satisfy \( N_2 \mid M \); this is possible because \( (q, N_2) = 1 \). This makes \( w \in \Gamma_0(N_2) \). As in Theorem 8.11 we have

\[
\Gamma_0(q^r)w P_{2,0}(\mathbb{Z}) = \bigcup_{u,t} \Gamma_0(q^r)wtu.
\]

Since \( (q, N_2) = 1 \), we can choose representatives \( u = u \left( \frac{a}{b} \right) \in \Gamma_0(N_2') \) by choosing \( N_2 \mid b \). And of course trivially \( t \in \Gamma_0(N_2) \). Letting \( \zeta_i \) be the various combinations of \( wtu \), and noting that \( wtu \in \Gamma_0(N_2') \), we are done.

\[\Box\]

Corollary 8.17. If \( q \nmid N_2 \), then \( \text{Sp}_2(\mathbb{Z}) = \Gamma_0(q^r)G_0(N_2) \).

Theorem 8.18. Let \( N = q_1^{e_1} \cdots q_r^{e_r} \) be the prime factorization of \( N \). Denote \( \psi_i = \Psi(q_i^{e_i}) \). For each \( q_i \), let \( w_{i,1}, \ldots, w_{i,\psi_i} \in \Gamma_0(N/q_i^{e_i}) \) be such that

\[
\text{Sp}_2(\mathbb{Z}) = \bigcup_{j=1}^{\psi_i} \Gamma_0(q_i^{e_i}) w_{i,j}.
\]

Then the disjoint coset decomposition of \( \Gamma_0(N)/\text{Sp}_2(\mathbb{Z}) \) is

\[
\text{Sp}_2(\mathbb{Z}) = \bigcup_{j_1=1}^{\psi_1} \cdots \bigcup_{j_n=1}^{\psi_n} \Gamma_0(N) w_{1,j_1} \cdots w_{n,j_n}.
\]

Proof. We remark that if \( H, K \) are subgroups of a group \( G \) such that \( G = HK \) and \( S \subseteq K \) is a set, then \( S \) forms a complete set of coset representatives of \( H \backslash G \) if and only if \( S \) forms a complete set of coset representatives of \( (H \cap K) \backslash K \). If we note

\[
\text{Sp}_2(\mathbb{Z}) \supset \Gamma_0(q_1^{e_1}) \supset \Gamma_0(q_1^{e_1} q_2^{e_2}) \supset \cdots \supset \Gamma_0(N),
\]

then the result follows from Corollary 8.17 and this remark.

Using Lemma 8.16 and Theorem 8.18 this Theorem 8.18 tells us how to construct coset representatives for \( \Gamma_0(N)/\text{Sp}_2(\mathbb{Z}) \). We now prove the results needed for the paramodular vanishing theorems.

Lemma 8.19. Let \( \alpha, N \in \mathbb{N} \) with \( \alpha \mid N \). Let \( w_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) and set \( \gamma = (\alpha, N/\alpha) \).

We have the disjoint single coset decomposition

\[
\Gamma_0(N)w_\alpha P_{2,0}(\mathbb{Z}) = \bigcup_{i=1}^{\kappa_\gamma} \Gamma_0(N) w_\alpha u_i.
\]
Lemma 8.20. Let the \( \kappa \) forces \( \alpha \). Theorem 8.11. The three key entries of \( wutw^{-1} \) are:

\[
(1, 2) : b - af \beta - bg\beta \\
(3, 2) : ce\beta \\
(4, 2) : d\beta - ac\beta - cf\beta^2 - dg\beta^2.
\]

If we assume \( wutw^{-1} \in \Gamma_0(q^*) \), then the \( (1, 2) \) entry forces \( \beta|b \). The \( (3, 2) \) entry forces \( \alpha|\gamma \) and the \( (4, 2) \) entry forces

\[
d - ae - cf\beta - dg\beta \equiv 0 \mod \alpha.
\]

Thus we have \( \gcd(\alpha, \beta)|(d - ae) \). Note that both \( \epsilon = \pm 1 \) are possible. Thus, if we let \( U_0 = \Gamma_0(\alpha, \beta) \), we have proven that \( wutw^{-1} \in \Gamma_0(q^*) \) implies \( u \in U_0 \). Furthermore, by Lemma 8.9 we have

\[
|U_0\setminus U| = \frac{\phi(\gamma)}{\psi(\gamma)} N \prod_{\text{prime } p|N} (1 + \frac{1}{p}).
\]

Now, for any \( u \in U_0 \), we will produce a \( t \in T \) such that \( wutw^{-1} \in \Gamma_0(N) \). Given such a \( u \), the key entry \( (3, 2) \) is already a multiple of \( N \). To make entry \( (4, 2) \) a multiple of \( N \), since \( d\beta^2 \) is relative prime to \( \frac{2}{\gamma} \gamma \), we just solve for \( g \) so that

\[
\frac{d - ae}{\gamma} - dt \beta\gamma \equiv 0 \mod \frac{\alpha}{\gamma}.
\]

To make entry \( (1, 2) \) a multiple of \( N \), since \( a \) is relative prime to \( \alpha \), we just solve for \( f \) so that \( \frac{b}{\gamma} - af - bg \) is a multiple of \( \alpha \). Thus the hypotheses of Lemma 8.8 are satisfied. We compute \( T_0 \) by noting that the key entries of \( wutw^{-1} \) are:

\[
(1, 2) : -f\beta \\
(3, 2) : 0 \\
(4, 2) : -g\beta^2
\]

and so \( t \in T_0 \) if and only if \( \alpha|f \) and \( \frac{2}{\gamma}|g \). Thus we have \( |T_0\setminus T| = \alpha \frac{2}{\gamma} \); further noting \( \kappa_\alpha = |T_0\setminus T| \cdot |U_0\setminus U| \) completes the proof.

Lemma 8.20. Let the \( \kappa_\alpha \) be as in Lemma 8.19. If \( \alpha_1 \not= \alpha_2 \), then \( \kappa_{\alpha_1} \not= \kappa_{\alpha_2} \).

Proof. Assume \( \kappa_{\alpha_1} = \kappa_{\alpha_2} \). Let \( \gamma_i = (\alpha_i, \frac{N}{\alpha_i}) \). Using the formula from Lemma 8.19

\[
\frac{\alpha_i^2}{\psi(\gamma_i)} \prod_{\text{prime } p|\gamma_i} (1 - \frac{1}{p}) = \frac{\alpha_i^2}{\psi(\gamma_2)} \prod_{\text{prime } p|\gamma_2} (1 - \frac{1}{p}),
\]

we can see that the largest odd prime \( q \) that divides \( N \) can only occur in the \( \alpha_i \) factors and possibly once in the denominator of each \( \prod_{p|\gamma_i} (1 - \frac{1}{p}) \). Thus this largest odd prime must occur to the same power \( r \) in each \( \alpha_1, \alpha_2 \), and hence also to the same power in each \( \gamma_i \). Because \( \psi \) is multiplicative, we can cancel \( \frac{\gamma^2}{\psi(\gamma)} (1 - \frac{1}{p}) \) from both sides, leaving us with the original relation with \( \alpha_i \) replaced by \( \frac{\alpha_i}{\gamma} \). In this way,
all the odd prime factors occur to the same power in \( \alpha_1, \alpha_2 \). We only have to take care of the case where the \( \alpha_i \) are both powers of 2. Let \( \alpha = 2^x \). As a function of \( x \), the formula for \( \kappa_\alpha \) is

\[
\kappa_\alpha = \frac{\alpha^2}{\psi(\gamma)} \prod_{\text{prime } p \mid \gamma} \left(1 - \frac{1}{p}\right) = \frac{2^{2x}}{\psi((2^x, N2^{-x}))}, \quad \begin{cases} 1 \text{ if } x = 0 \\ \frac{1}{2} \text{ if } x > 0 \end{cases}
\]

Call the above expression \( F(x) \). Note \( F(0) = 1 \) and \( F(1) = 2 \). Increasing a positive integer \( x \) by 1 increases \( F(x) \) by at least a factor of 2 because \( \psi((2^x, N2^{-x})) \) can increase at most by a factor of 2 whereas \( 2^{2x} \) increases by a factor of 4. This completes the proof. \( \square \)

9. Appendix

This appendix makes a correction to [31]. On page 452, the paragraph that begins “An immediate virtue . . .” is not correct in the generality it is written. It is correct for degree two paramodular forms of even weight and, with this restriction, it applies to all the examples in the article. The issue is that the global \( \tilde{\Phi} \) image of a Siegel modular form of trivial character and odd weight may have some components with nontrivial character. For example, we have

\[
\tilde{\Phi} : M_k(K(9)) \to M_k(SL_2(\mathbb{Z})) \oplus M_k(\tilde{\Gamma}_0(3), \chi) \oplus M_k(SL_2(\mathbb{Z})),
\]

where \( \tilde{\Gamma}_0(3) = \langle \Gamma_0(3), -I_2 \rangle \) and \( \chi : \tilde{\Gamma}_0(3) \to \{ \pm 1 \} \) is the character trivial on \( \Gamma_0(3) \) but satisfying \( \chi(-I_2) = (-1)^k \). We thank John Duncan for discussions about paramodular forms of odd weight.

References

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