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IRREDUCIBLE NON-CUSPIDAL
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BY

Dr. Ralf Schmidt, Chair

Dr. Andy Magid

Dr. Tomasz Przebinda

Dr. Alan Roche

Dr. Deborah Trytten

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Abstract

Admissible non–supercuspidal representations of $\mathrm{GSp}(4, F)$, where F is a local field of characteristic zero with an odd-ordered residue field $\mathbb{F}_q$, have finite dimensional spaces of fixed vectors under the action of principal congruence subgroups. We can say precisely what these dimensions are for nearly all local fields and principal congruence subgroups of level $\mathfrak{p}$ by understanding the non–cuspidal representation theory of the finite group $\mathrm{GSp}(4, \mathbb{F}_q)$. The conjugacy classes and the list of irreducible characters of this group are given. Genericity and cuspidality of the irreducible characters are also determined.

Chapter 1

Introduction

Representations of linear groups over a finite or a local field F are constructed from cuspidal representations, as noted in [8] by Harish-Chandra, in a way analogous to the construction of Eisenstein series from cusp forms. The representation theory of these groups can be thought of as having two kinds of representations: cuspidal and non–cuspidal.

For reductive algebraic groups over F, non–cuspidal representations are constructed from cuspidal representations by parabolic induction. One parabolically induces a cuspidal representation defined on a parabolic subgroup and then decomposes the induced representation into irreducible constituents. When all of the parabolic subgroups and cuspidal representations of those subgroups have been exhausted, the result is a list of every irreducible non–cuspidal representation of the group. The only irreducible representations of the group remaining are those that cannot be obtained through parabolic induction. These are the cuspidal representations.

Let G be a connected algebraic group over a finite field F. A maximal Zariski-connected solvable algebraic subgroup of G is called a *Borel subgroup* of G. Let B, B' be Borel subgroups. Then $B' = gBg^{-1}$ for some $g \in G$. A *parabolic subgroup* P

of G is a subgroup that contains the Borel subgroup (or one of its conjugates) as a subgroup. A *Levi subgroup* M of P is a maximal reductive subgroup (determined up to conjugacy) of P. The *unipotent radical* U of P is its maximal unipotent subgroup.

The group we are interested in is the general symplectic group $\mathrm{GSp}(4, \mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ of odd characteristic. We look at cuspidal representations defined on the *Borel subgroup* consisting of upper triangular matrices and two parabolic subgroups called the *Siegel parabolic* and the *Klingen parabolic*, which are each block upper triangular. Parabolically inducing cuspidal representations of these three subgroups turns out to yield the complete collection of non–cuspidal representations of $\mathrm{GSp}(4, \mathbb{F}_q)$. Character theory is used to determine the non–cuspidal representations by decomposing parabolically induced cuspidal representations into irreducible constituents.

Our first step is to find the list of conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$. This list is used to compute the classes of the Borel, the Siegel parabolic, and the Klingen parabolic subgroups. The main tool we use for determining the conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$ is a paper of Wall [15], which was also used to determine the conjugacy classes of the symplectic group $\mathrm{Sp}(4, \mathbb{F}_q)$ in Srinivasan's paper [14]. An isomorphism between $\mathrm{GSp}(4, \mathbb{F}_q)$ modulo its center and a special orthogonal group is used to solve the conjugacy class problem for $\mathrm{GSp}(4, \mathbb{F}_q)$. We then determine how the conjugacy classes split in the Borel, the Siegel parabolic, and the Klingen parabolic subgroups.

All of the irreducible characters of the finite group $\mathrm{GSp}(4, \mathbb{F}_q)$ and their cuspidality and genericity are determined. Cuspidality is determined by defining cuspidal representations on the Borel, the Siegel parabolic, and the Klingen parabolic subgroup and then inducing. The irreducible non–cuspidal representations are precisely the irreducible constituents of these induced representation. Criteria are determined for these induced characters to be irreducible. If an induced character is reducible, then the constituents are found.

Before doing the computations, we already have some idea as to when the induced character is irreducible and what the irreducible constituents are if it is reducible. It is expected that the results will be similar to those in Sally and Tadić's paper [12]. In [12], the irreducible non–supercuspidal representations are given for $\mathrm{GSp}(4, F)$, where F is a non–archimedean local field of characteristic 0. These results are summarized in a table given in [11]. In the local field case, the terms supercuspidal and non–supercuspidal are normally used instead of cuspidal and non–cuspidal. The natural analogous representation in the finite field case turns out to be correct and is verified by computing the character values. However, some of the results in [12] don't have a clear analogue in the finite field case.

1.1 Basic definitions and notations

Let $\mathbb{F}_q$ denote the finite field with $q = p^n$ elements, with p an odd prime.

Definition 1.1.1. The group $G = \mathrm{GSp}(4, \mathbb{F}_q)$ is defined as

$$\mathrm{GSp}(4, \mathbb{F}_q) := \{g \in \mathrm{GL}(4, \mathbb{F}_q) : {}^t g J g = \lambda J\}, \text{ where } J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \text{-1} & & \\ \text{-1} & & & \end{pmatrix}$$

for some $\lambda \in \mathbb{F}_q^\times$, which will be denoted by $\lambda(g)$ and called the *multiplier* of g. The set of all $g \in \mathrm{GSp}(4, \mathbb{F}_q)$ such that $\lambda(g) = 1$ is a subgroup and is denoted by $\mathrm{Sp}(4, \mathbb{F}_q)$.

Note that for any $g \in G$, we can uniquely write g as

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \lambda(g) & \\ & & & \lambda(g) \end{pmatrix} \cdot g',$$

where $g' \in \mathrm{Sp}(4, \mathbb{F}_q)$.

The order of $\mathrm{Sp}(4, \mathbb{F}_q)$, as computed by Wall, is $q^4(q^4-1)(q^2-1)$. So the order of $\mathrm{GSp}(4, \mathbb{F}_q)$ is $q^4(q^4-1)(q^2-1)(q-1)$.

We recall some basic definitions in representation theory. By a group, we mean a finite group and by a vector space, we mean a finite-dimensional complex vector space.

Definition 1.1.2. A *representation* (π, V) of a group G is a group homomorphism

$$\pi : G \longrightarrow \mathrm{G}L(V)$$

and a vector space V, where $\mathrm{GL}(V)$ is the group of all invertible linear automorphisms of V. The representation (π, V) will be referred to by either π or V. It is also said that the group G acts on V by the action of π. The *dimension of the representation* π is defined as the dimension of the vector space V. A *linear representation* is a one-dimensional representation.

Definition 1.1.3. Two representations (π, V) and (π', V') of a group G are called *equivalent* or *isomorphic* if there exists an invertible map $h : V \longrightarrow V'$ such that for all $g \in G$

$$\pi(g) = h^{-1}\pi'(g)h.$$

Definition 1.1.4. Let π be a representation of the group G. A *subrepresentation* of π is the restriction of the action of π to a subspace $U \subseteq V$ such that U is invariant under the action π.

Definition 1.1.5. A representation is called *irreducible* if there is no nontrivial invariant subspace. The set of all equivalence classes of irreducible representations of a group G is denoted by $\mathrm{Irr}(G)$.

Given a representation V of a group G, a representation of any subgroup H of G is obtained simply by restricting the representation to H, denoted by $\mathrm{Res}_H^G V$ or by $\mathrm{Res} V$ when the group G and subgroup H are clear from context. The vector space V in these notations may be replaced by the action π of the representation. When V is replaced by its character χ, $\mathrm{Res}_H^G \chi$ or $\mathrm{Res}\chi$ will denote the character of the restricted representation.

A representation of G can be obtained from a representation on a subgroup H of G through the process of *induction*. The representation of G induced from a representation V of H is denoted by $\mathrm{Ind}_H^G V$, or, if the group G and subgroup H are clear from context, by $\mathrm{Ind} V$. Again, the vector space V in these notations may be replaced by the action π of the representation and when V is replaced by its character χ, $\mathrm{Ind}_H^G \chi$ or $\mathrm{Ind}\chi$ will denote the character of the induced representation.

The induced representation $\mathrm{Ind}_H^G V$ can be realized by the following construction. Let G be a group and H a subgroup of G. Let (π, V) be a representation of H. The induced representation $\mathrm{Ind}_H^G V$ is isomorphic to the space of functions V^G given by

$$V^G = \{f : G \to V : f(hg) = \pi(h)f(g), \text{for } h \in H, g \in G\},$$

with the group G acting on this space by right translation.

1.2 Character theory

When the representation (π, V) is finite dimensional, the group $\mathrm{GL}(V)$ can be viewed as the group of invertible $n \times n$ matrices, where n is the dimension of V. Under this identification, the trace of $\pi(g)$ is defined for any $g \in G$.

Definition 1.2.1. Let π be a finite dimensional representation of a group G. The *character of* π is a function $\chi : G \longrightarrow \mathbb{C}^*$ defined by

$$\chi(g) = \mathrm{tr}(\pi(g)),$$

where tr denotes the trace map.

It follows from properties of the trace map that characters are *class functions*, i.e., they are constant on conjugacy classes. Note that for the identity element $I \in G$, $\chi(I) = \mathrm{tr}(\pi(I))$ is the trace of the identity map of the space of π, which is precisely its dimension.

The character χ also takes the same values on equivalent representations. Indeed, for two equivalent representations π and π' of G, we have, for some invertible map h from the space of π to the space of π', $\pi(g) = h^{-1}\pi(g)h$ for all $g \in G$ and

$$\mathrm{tr}(\pi(g)) = \mathrm{tr}(h^{-1}\pi'(g)h) = \mathrm{tr}(\pi'(g))$$

for all $g \in G$.

Definition 1.2.2. Let G be a group. The *character table of* G is a square array of complex numbers with rows indexed by the inequivalent irreducible characters of G and the columns indexed by the conjugacy classes. The entry in row χ and column C is the value of χ on the conjugacy class C.

	C_1	C_2	$\cdots$	C_n
χ_1	$\chi_1(C_1)$	$\chi_1(C_2)$	$\cdots$	$\chi_1(C_n)$
χ_2	$\chi_2(C_1)$	$\chi_2(C_2)$	$\cdots$	$\chi_2(C_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_n	$\chi_n(C_1)$	$\chi_n(C_2)$	$\cdots$	$\chi_n(C_n)$

For a particular character χ of G, the *character table of* χ is defined to be a table of complex numbers with rows indexed by the conjugacy classes of G with the entry in row C denoting the value of χ on the conjugacy class C.

C_1	$\chi(C_1)$
C_2	$\chi(C_2)$
$\vdots$	$\vdots$
C_n	$\chi_n(C_n)$

Definition 1.2.3. Let χ_1 and χ_2 be characters of a group G. The *inner product* of χ_1 and χ_2 is defined as

$$(\chi_1, \chi_2)_G = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}$$

The subscript G will be dropped from the notation of the inner product $(,)_G$ when the group G is clear from context.

Theorem 1.2.4. *Let G be a group. The irreducible characters of G form an orthonormal basis for the vector space of all class functions of G with respect to the inner product $(\cdot, \cdot)$.*

Corollary 1.2.5. *Let χ be a character of a group G. Then χ is irreducible if and only if $(\chi, \chi) = 1$.*

When a representation π of a subgroup H is induced to the group G, the induced character χ^G can be determined using the character χ of π in the following way.

$$\mathrm{Ind}_H^G(\chi)(g) = \chi^G(g) = \frac{1}{|H|} \sum_{x \in G,\, xgx^{-1} \in H} \chi(xgx^{-1}), \text{ for } g \in G.$$

As given in [4], the induced character's values on conjugacy classes of the group G can also be found. Let C be a conjugacy class of a group G. Then the conjugacy class C splits into distinct conjugacy classes of a subgroup H, say $C = D_1 \sqcup \ldots \sqcup D_r$. The value of the induced character is given by the formula

$$\mathrm{Ind}_H^G(\chi)(C) = \chi^G(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi(D_i).$$

Another important standard result is the following.

Lemma 1.2.6. ***(Frobenius Reciprocity)*** *Let $H \leq G$ and let χ be a character of G and ψ be a character of H. Then*

$$(\chi, \mathrm{Ind}_H^G \psi)_G = (\mathrm{Res}_H^G \chi, \psi)_H$$

An irreducible character on a subgroup H might not retain its irreducibility when induced. It is important to note that finite groups have the *complete reducibility property*, i.e., every representation of the group decomposes into a direct sum of irreducible representations. A representation (π, V) of a finite group G is said to be *multiplicity free* if in its decomposition into irreducibles, no irreducible representation occurs more than once.

By Schur's Lemma, the center of G acts by scalars in an irreducible representation (π, V). In particular, if $G = \mathrm{GL}(n, \mathbb{F}_q)$ or $G = \mathrm{GSp}(4, \mathbb{F}_q)$, then the center of G consists of non-zero scalar multiples of the identity matrix. The center of G is isomorphic to $\mathbb{F}_q^\times$ so there exists a character $\omega_\pi : \mathbb{F}_q^\times \longrightarrow \mathbb{C}^\times$, known as the *central character of* π, such that

$$\pi \begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} v = \omega_\pi(z) v$$

for every $z \in \mathbb{F}_q^\times, v \in V$.

If χ is a character of $\mathbb{F}_q^\times$ and (π, V) is a representation of $\mathrm{GSp}(4, \mathbb{F}_q)$, then a new representation on V can be defined by

$$(\chi\pi)(g) = \chi(\lambda(g))\pi(g),$$

where $\lambda(g)$ is the multiplier of g. This representation is denoted by $\chi\pi$ and called the *twist* of the representation π by the character χ. If ω_π is the central character of π, then the central character of $\chi\pi$ is $\omega_\pi\chi^2$.

1.3 Representations of $\mathrm{GL}(2, \mathbb{F}_q)$

The representation theory of $\mathrm{GL}(2, \mathbb{F}_q)$ is the first object of study in order to understand the irreducible non–cuspidal representations of $\mathrm{GSp}(4, \mathbb{F}_q)$. A nice treatment of the representation theory of $\mathrm{GL}(2, \mathbb{F}_q)$ is given in [1] and in [4]. The methods used to study the representation theory of $\mathrm{GL}(2, \mathbb{F}_q)$ in [1] can be extended with some modification to study the representation theory of $\mathrm{GL}(2, \mathbb{F})$, where $\mathbb{F}$ is a local field.

Let $B_{\mathrm{GL}(2)}$ be the subgroup of $\mathrm{GL}(2, \mathbb{F}_q)$ consisting of all upper triangular matrices. $B_{\mathrm{GL}(2)}$ is called the *Borel subgroup* of $\mathrm{GL}(2, \mathbb{F}_q)$.

$$B_{\mathrm{GL}(2)} = \left\{ \begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_q) \right\}.$$

The subgroup of $B_{\mathrm{GL}(2)}$ consisting of matrices with diagonal entries equal to 1 will be denoted by $N_{\mathrm{GL}(2)}$ and the subgroup of $B_{\mathrm{GL}(2)}$ consisting of diagonal matrices will be denoted by $T_{\mathrm{GL}(2)}$.

Let χ_1, χ_2 be characters of $\mathbb{F}_q^\times$. Define a character χ of $B_{\mathrm{GL}(2)}$ by

$$\chi \begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} = \chi_1(y_1)\chi_2(y_2).$$

Denote the representation of $\mathrm{GL}(2, \mathbb{F}_q)$ induced from this character of $B_{\mathrm{GL}(2)}$ by $\chi_1 \times \chi_2$.

Theorem 1.3.1. *Let χ_1, χ_2, μ_1 and μ_2 be characters of $\mathbb{F}_q^\times$. Then $\chi_1 \times \chi_2$ is an irreducible representation of degree $q+1$ of $\mathrm{GL}(2, \mathbb{F}_q)$ unless $\chi_1 = \chi_2$, in which case it is the direct sum of two irreducible representations having degrees 1 and q. We have*

$$\chi_1 \times \chi_2 \cong \mu_1 \times \mu_2$$

if and only if either

$$\chi_1 = \mu_1 \text{ and } \chi_2 = \mu_2$$

or

$$\chi_1 = \mu_2 \text{ and } \chi_2 = \mu_1.$$

Proof: See [1].

The irreducible representation of dimension 1 contained in $\chi \times \chi$ is the character $g \mapsto \chi(\det(g))$. The other q-dimensional representation is obtained by taking the tensor of this character with the q-dimensional irreducible subrepresentation of $1_{\mathbb{F}_q^\times} \times 1_{\mathbb{F}_q^\times}$, where $1_{\mathbb{F}_q^\times}$ denotes the trivial character of $\mathbb{F}_q^\times$. This q-dimensional representation is called the *Steinberg representation of* $\mathrm{GL}(2, \mathbb{F}_q)$. The Steinberg representation of $\mathrm{GL}(2, \mathbb{F}_q)$ will play a role in the representation theory of $\mathrm{GSp}(4, \mathbb{F}_q)$. The irreducible representations $\chi_1 \times \chi_2$ are called *representations of the principal series.*

A representation (π, V) of $\mathrm{GL}(2, \mathbb{F}_q)$ is said to be *cuspidal* if there does not exist a non-zero linear functional l on V such that

$$l\left(\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}v\right) = l(v)$$

for all $v \in V, x \in \mathbb{F}_q$. If F is a local field and if (π, V) is a representation of $\mathrm{GL}(2, F)$ such that there does not exist a non-zero linear functional l on V satisfying the condition above, then the representation (π, V) is said to be *supercuspidal*.

Proposition 1.3.2. *Let (π, V) be a cuspidal representation of* $\mathrm{GL}(2, \mathbb{F}_q)$*. Then the dimension of V is a multiple of $q - 1$.*

Proof: See [1].

The irreducible cuspidal representations of $\mathrm{GL}(2, \mathbb{F}_q)$ and their character tables are found in [4]. Let γ be a generator of $\mathbb{F}_q^\times$ and let ϕ be a character of $\mathbb{F}_{q^2}^\times$ such that $\phi \neq \phi^q$. All irreducible cuspidal representations of $\mathrm{GL}(2, \mathbb{F}_q)$ are of the form X_ϕ where the character table of X_ϕ is given by the table below.

Table 1.1: X_ϕ character values

Conjugacy class	X_ϕ **character value**
$a_x = \begin{pmatrix}x & \\ & x\end{pmatrix}$	$(q-1)\phi(x)$
$b_x = \begin{pmatrix}x & 1\\ & x\end{pmatrix}$	$-\phi(x)$
$c_{x,y} = \begin{pmatrix}x & \\ & y\end{pmatrix}$	0

Table 1.1 – Continued

Conjugacy class	X_ϕ **character value**
$d_{x,y} = \begin{pmatrix} x & y\gamma \\ y & x \end{pmatrix}$	$-\left(\phi(x+y\sqrt{\gamma})+\phi(x-y\sqrt{\gamma})\right)$

Let ψ be a non-trivial character of $\mathbb{F}_q$ and let ψ_N be the character of $N_{\mathrm{GL}(2)}$ defined by

$$\psi_N \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \psi(x).$$

This defines a representation of $N_{\mathrm{GL}(2)}$. The representation of $\mathrm{GL}(2,\mathbb{F}_q)$ induced from ψ_N is denoted by $\mathcal{G}$.

Theorem 1.3.3. ***(Uniqueness of Whittaker models)*** *The representation $\mathcal{G}$ is multiplicity free. Every irreducible representation of* $\mathrm{GL}(2,\mathbb{F}_q)$ *that is not one dimensional occurs in $\mathcal{G}$ with multiplicity precisely 1.*

Proof: See [1].

If (π, V) is an irreducible representation that can be embedded into $\mathcal{G}$, we call its image a *Whittaker model of* π. A Whittaker model of π is then a space $\mathcal{W}(\pi)$ of functions $W : \mathrm{GL}(2,\mathbb{F}_q) \longrightarrow \mathbb{C}$ having the property that

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x)W(g).$$

The functions W are invariant under right translation and give a representation of $\mathrm{GL}(2,\mathbb{F}_q)$ which is isomorphic to π. Representations that have a Whittaker model are called *generic*.

1.4 Generic representations of $\mathrm{GSp}(4, \mathbb{F}_q)$

Whittaker models for representations of $\mathrm{GSp}(4, \mathbb{F}_q)$ can also be defined. Consider the subgroup $N_{\mathrm{GSp}(4)}$ of $\mathrm{GSp}(4, \mathbb{F}_q)$ defined as

$$N_{\mathrm{GSp}(4)} = \left\{ \begin{pmatrix} 1 & y & * & * \\ & 1 & x & * \\ & & 1 & -y \\ & & & 1 \end{pmatrix} \in \mathrm{GSp}(4, \mathbb{F}_q) \right\}.$$

Let ψ_1 and ψ_2 be non-trivial characters of $\mathbb{F}_q$ and let ψ_N be the character of $N_{\mathrm{GSp}(4)}$ defined by

$$\psi_N \begin{pmatrix} 1 & y & * & * \\ & 1 & x & * \\ & & 1 & -y \\ & & & 1 \end{pmatrix} = \psi_1(x)\psi_2(y).$$

This defines a representation of $N_{\mathrm{GSp}(4)}$. Denote the representation of $\mathrm{GSp}(4, \mathbb{F}_q)$ induced from ψ_N by $\mathcal{G}$ and its character by $\chi_{\mathcal{G}}$.

If (π, V) is an irreducible representation that can be embedded into $\mathcal{G}$, we call its image a *Whittaker model of* π and say that π is *generic*. A Whittaker model of π is then a space $\mathcal{W}(\pi)$ of functions $W : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow \mathbb{C}$ having the property that

$$W\left(\begin{pmatrix} 1 & y & * & * \\ & 1 & x & * \\ & & 1 & -y \\ & & & 1 \end{pmatrix} g \right) = \psi_1(x)\psi_2(y)W(g).$$

Genericity of an irreducible representation π of $\mathrm{GSp}(4, \mathbb{F}_q)$ is easy to determine using character theory. Indeed, for a particular irreducible non–cuspidal representation π of $\mathrm{GSp}(4, \mathbb{F}_q)$ with character χ_π, one computes the inner product $(\chi_\pi, \chi_{\mathcal{G}})$. If $(\chi_\pi, \chi_{\mathcal{G}}) = 0$, then π is not generic. If $(\chi_\pi, \chi_{\mathcal{G}}) \neq 0$, then π is generic. Moreover, when $(\chi_\pi, \chi_{\mathcal{G}}) \neq 0$, then $(\chi_\pi, \chi_{\mathcal{G}}) = 1$, i.e., Whittaker models are unique. The uniqueness of Whittaker models is known in general, but it is verified computationally.

Therefore, to determine genericity, we compute the conjugacy classes of $N_{\mathrm{GSp}(4)}$ and the character table of $\mathcal{G}$. Note that the character of the representation of $N_{\mathrm{GSp}(4)}$ defined above is $\psi_1(x)\psi_2(y)$.

Chapter 2

Conjugacy classes

The conjugacy classes and their orders of the unitary, symplectic and orthogonal groups can be determined using the results of Wall [15]. Srinivasan, in [14], used Wall's results to explicitly determine the conjugacy classes and orders of centralizers of elements of $\mathrm{Sp}(4, \mathbb{F}_q)$.

Wall's results cannot be directly used to determine the conjugacy classes of the group $\mathrm{GSp}(4, \mathbb{F}_q)$ but they can be used to find the classes of $\mathrm{SO}(5, \mathbb{F}_q)$. This is particularly useful because $\mathrm{SO}(5, \mathbb{F}_q)$ is isomorphic to $\mathrm{PGSp}(4, \mathbb{F}_q) := \mathrm{GSp}(4, \mathbb{F}_q)/Z$, where Z is the center of $\mathrm{GSp}(4, \mathbb{F}_q)$. These classes are then used to determine the conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$.

2.1 An isomorphism from $\mathrm{PGSp}(4, \mathbb{F}_q)$ to $\mathrm{SO}(5, \mathbb{F}_q)$

We follow the method in [11] to define an isomorphism.

Definition 2.1.1. $\mathrm{O}(5, \mathbb{F}_q)'$ is defined by $\{g \in \mathrm{GL}(5, \mathbb{F}_q) : {}^t g g = I_5\}$, where I_5 is the 5×5 identity matrix and $\mathrm{SO}(5, \mathbb{F}_q)' := \{g \in \mathrm{O}(5, \mathbb{F}_q)' : det(g) = 1\}$. Define

$$\mathrm{O}(5,\mathbb{F}_q) := \{g \in \mathrm{GL}(5,\mathbb{F}_q) : {}^t g J_5 g = J_5\}, \text{ where } J_5 = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

and $\mathrm{SO}(5,\mathbb{F}_q) := \{g \in \mathrm{O}(5,\mathbb{F}_q) : det(g) = 1\}$.

We will show $\mathrm{PGSp}(4,\mathbb{F}_q) \cong \mathrm{SO}(5,\mathbb{F}_q)'$ first by showing that $\mathrm{PGSp}(4,\mathbb{F}_q)$ is isomorphic to $\mathrm{SO}(5,\mathbb{F}_q)$, then showing that $\mathrm{SO}(5,\mathbb{F}_q)$ is isomorphic to $\mathrm{SO}(5,\mathbb{F}_q)'$. First note that the center of $\mathrm{GSp}(4,\mathbb{F}_q)$ consists of diagonal matrices, with non-zero entries on the diagonal. Recall that the characteristic of the field $\mathbb{F}_q$ is $p \neq 2$.

Let $V = \mathbb{F}_q^4$ and e_1, e_2, e_3, e_4 be the standard basis vectors of V. The group $\mathrm{GSp}(4,\mathbb{F}_q)$ acts on V by matrix multiplication from the left. $\mathrm{GSp}(4,\mathbb{F}_q)$ also acts on the tensor $V \otimes V$ twisted with the inverse of the multiplier. So

$$\rho(g)(v \otimes w) = \lambda(g)^{-1}(gv) \otimes (gw).$$

The action ρ of $\mathrm{GSp}(4,\mathbb{F}_q)$ on $V \otimes V$ is trivial on the center of $\mathrm{GSp}(4,\mathbb{F}_q)$ and so there is an action of $\mathrm{PGSp}(4,\mathbb{F}_q)$ on $V \otimes V$.

Define a symplectic form on V by

$$(v, v') := {}^t v \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} v',$$

for v, $v' \in V$. Now define a symmetric bilinear form on $V \otimes V$, given on pure tensors by

$$\langle v \otimes w, v' \otimes w' \rangle := (v, v')(w, w'),$$

for $v \otimes w,\ v' \otimes w' \in V \otimes V$.

Both of these bilinear forms are invariant under the action of $\mathrm{Sp}(4, \mathbb{F}_q)$ and the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is preserved by the action ρ of $\mathrm{GSp}(4, \mathbb{F}_q)$. We embed $V \wedge V$ in $V \otimes V$ by the map $v \wedge w \longmapsto \frac{1}{2}(v \otimes w - w \otimes v)$.

The restriction of our symmetric bilinear form $\langle \cdot, \cdot \rangle$ to the wedge $V \wedge V$ is given by

$$\langle v \wedge w, v' \wedge w' \rangle = \frac{1}{2}((v, v')(w, w') - (v, w')(w, v')).$$

Let X be the image of the 5-dimensional subspace spanned by

$$e_1 \wedge e_2, \qquad 2e_1 \wedge e_3, \qquad e_1 \wedge e_4 - e_2 \wedge e_3, \qquad e_2 \wedge e_4, \qquad 2e_4 \wedge e_3.$$

Explicitly, X is spanned by the vectors

$$\begin{aligned}
x_1 &= \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1),\\
x_2 &= e_1 \otimes e_3 - e_3 \otimes e_1,\\
x_3 &= \frac{1}{2}(e_1 \otimes e_4 - e_4 \otimes e_1 - e_2 \otimes e_3 + e_3 \otimes e_2),\\
x_4 &= \frac{1}{2}(e_2 \otimes e_4 - e_4 \otimes e_2),\\
x_5 &= e_4 \otimes e_3 - e_3 \otimes e_4.
\end{aligned}$$

It is straightforward to check that the matrix of $\langle \cdot, \cdot \rangle$ with respect to this basis is J_5 and that X is invariant under the action ρ of $\mathrm{GSp}(4, \mathbb{F}_q)$. Since $\langle \cdot, \cdot \rangle$ is preserved by this action so there is a homomorphism

$$\rho_5 : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow \mathrm{SO}(5, F_q), \qquad \rho_5(g) := (a_{ij})$$

where the a_{ij} are determined by the action ρ on $V \otimes V$, i.e.,

$$\rho(g)x_j = a_{1j}x_1 + a_{2j}x_2 + a_{3j}x_3 + a_{4j}x_4 + a_{5j}x_5.$$

The kernel of ρ_5 is the center Z of $\mathrm{GSp}(4, \mathbb{F}_q)$ and so there is an isomorphism

$$\rho_5 : PGSp(4, \mathbb{F}_q) \xrightarrow{\sim} \mathrm{SO}(5, F_q)$$

We will now give an isomorphism from $\mathrm{SO}(5, \mathbb{F}_q)$ to $\mathrm{SO}(5, \mathbb{F}_q)'$. First note that J_5 is the matrix of a non-degenerate symmetric bilinear form. Define the quadratic form

$$Q'(x) := \langle x, x \rangle = {}^t x J_5 x,$$

where $\langle \cdot, \cdot \rangle$ is the form defined above. This form is equivalent to the bilinear form

$$Q(x) := {}^t x x = I_5,$$

i.e., there exists a P such that $P J_5 {}^t P = I_5$. To see this, choose a new basis for X such the matrix of the bilinear form is I_5. For $x \in X$, x is of the form

$$x = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_5 x_5.$$

Then,

$$Q'(x) = 2x_1 x_5 + 2x_2 x_4 + x_3^2.$$

Define a new basis v_1, v_2, v_3, v_4, v_5 of X as follows.

$$v_1 = \frac{1}{2}x_1 + x_5, \qquad v_2 = \frac{1}{2}x_2 + x_4, \qquad v_3 = x_3.$$

To choose v_4, v_5, we look at two cases.

Case 1: $p \equiv 1 \pmod 4$

Then -1 is a square. Say $b^2 = -1$ for some $b \in \mathbb{F}_q$. Then we choose

$$v_4 = b(\frac{1}{2}x_1 - x_5),\ v_5 = b(\frac{1}{2}x_2 - x_4).$$

Case 2: $p \equiv 3 \pmod 4$

Over a finite field, -1 is the sum of two squares, say $a_1^2 + a_2^2 = -1$. Choose

$$v_4 = a_1(\frac{1}{2}x_1 - x_5) + a_2(\frac{1}{2}x_2 - x_4),\ v_5 = a_1(\frac{1}{2}x_1 - x_5) - a_2(\frac{1}{2}x_2 - x_4).$$

With this new basis, it is clear that an equivalent non-degenerate symmetric bilinear form of J_5 is I_5. Define the matrices

$$P_1 = \begin{pmatrix} 1 & & & & \frac{1}{2} \\ & 1 & & \frac{1}{2} & \\ & & 1 & & \\ -b & & & & \frac{b}{2} \\ & -b & & \frac{b}{2} & \end{pmatrix},\ P_2 = \begin{pmatrix} 1 & & & & \frac{1}{2} \\ & 1 & & \frac{1}{2} & \\ & & 1 & & \\ a_1 & a_2 & & -\frac{a_2}{2} & -\frac{a_1}{2} \\ a_2 & -a_1 & & \frac{a_1}{2} & -\frac{a_2}{2} \end{pmatrix}.$$

In either case, $P_i J_5 \,{}^t P_i = I_5$. Let $P = P_i$, for the appropriate P_i. Define the map

$$\phi_P : \mathrm{SO}(5, F_q)' \longrightarrow \mathrm{SO}(5, \mathbb{F}_q)$$

$$\phi_P(g) := {}^t P^{-1} g \,{}^t P,$$

and ϕ_P is an isomorphism.

The maps defined above are composed to get an isomorphism from $\mathrm{GSp}(4, \mathbb{F}_q)$ to $\mathrm{SO}(5, \mathbb{F}_q)'$, which will be called $\rho_{5,P} := \phi_P \circ \rho_5$. The conjugacy classes of $\mathrm{PGSp}(4, \mathbb{F}_q)$ can now be determined by computing either the conjugacy classes of $\mathrm{SO}(5, \mathbb{F}_q)$ or of

$\mathrm{SO}(5,\mathbb{F}_q)'$, then using the appropriate isomorphism.

2.2 $\mathrm{SO}(5,\mathbb{F}_q)$ and $\mathrm{PGSp}(4,\mathbb{F}_q)$

We can determine the conjugacy classes of $\mathrm{O}(5,\mathbb{F}_q)$ using Wall [15]. To do this, we first find the Jordan canonical forms of elements of $\mathrm{GL}(5,\mathbb{F}_q)$ whose conjugacy class has a nonempty intersection with $\mathrm{O}(5,\mathbb{F}_q)$. Once the possible Jordan canonical forms of elements in $\mathrm{O}(5,\mathbb{F}_q)$ are found, it is straightforward to find class representatives in $\mathrm{O}(5,\mathbb{F}_q)$. Wall [15] also gives a formula for the number of conjugacy classes of $\mathrm{O}(5,\mathbb{F}_q)$, which is determined to be $2q^2+6q+14$. Once the complete list of the classes of $\mathrm{O}(5,\mathbb{F}_q)$ is computed, the class representatives and orders of centralizers for $\mathrm{SO}(5,\mathbb{F}_q)$ are easily found.

Let κ be a generator of $\mathbb{F}_{q^4}^\times$ and let $\zeta=\kappa^{q^2-1}$, $\theta=\kappa^{q^2+1}$, $\eta=\theta^{q-1}$, and $\gamma=\theta^{q+1}$. The element η is the generator of the set of elements in $\mathbb{F}_{q^2}$ whose norm over $\mathbb{F}_q$ is 1 and ζ is the generator of the set of elements in $\mathbb{F}_{q^4}$ whose norm over $\mathbb{F}_{q^2}$ is 1. Let $a,b\in\mathbb{F}_q^\times$ be such that $-a^2+b^2\gamma$ is a square. Let $c=\gamma+1$.

Define the sets

$R_1=\{1,...,\frac{1}{4}(q^2-1)\}$,

R_2 is a set of $\frac{1}{4}(q-1)^2$ distinct positive integers i such that θ^i, θ^{-i}, θ^{qi}, and θ^{-qi} are all distinct,

$T_1=\{1,...,\frac{1}{2}(q-3)\}$,

$T_2=\{1,...,\frac{1}{2}(q-1)\}$,

$T_3=\{1,...,q-1\}$.

The table below lists all of the conjugacy classes of $\mathrm{SO}(5,\mathbb{F}_q)$ and $\mathrm{PGSp}(4,\mathbb{F}_q)$ together with the order of their centralizers in each group. Note that the class representatives are written in a form that may not be in $\mathrm{SO}(5,\mathbb{F}_q)$, respectively $\mathrm{PGSp}(4,\mathbb{F}_q)$, but will belong to $\mathrm{SO}(5, \mathbb{F}_{q^4})$, respectively $\mathrm{PGSp}(4, \mathbb{F}_{q^4})$. This is done to indicate their eigenvalues, which are used to determine the orders of their centralizers.

Table 2.1: Conjugacy classes of $\mathrm{SO}(5, \mathbb{F}_q)$ and $\mathrm{PGSp}(4, \mathbb{F}_q)$

Notation	**Class representative in** $\mathrm{SO}(5, \mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4, \mathbb{F}_q)$	**Order of centralizer**
A_1	$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$q^4(q^4-1)(q^2-1)$
A_2	$\begin{pmatrix} 1 & 2 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -2 \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$q^4(q^2-1)$
A_{31}	$\begin{pmatrix} 1 & -2 & & -1 & -2 \\ & 1 & & & 1 \\ & & 1 & & \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & 1 \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$2q^3(q-1)$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5, \mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4, \mathbb{F}_q)$	**Order of centralizer**
A_{32}	$\begin{pmatrix} 1 & -2 & & -\gamma & -2\gamma \\ & 1 & & & \gamma \\ & & 1 & & \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \gamma \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$2q^3(q+1)$
A_5	$\begin{pmatrix} 1 & -2 & & & \\ & 1 & -1 & -\frac{1}{2} & -1 \\ & & 1 & 1 & 2 \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & -1 & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}$	q^2
B_{11}	$\begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$	$2q^2(q^2-1)^2$
B_{12}	$\begin{pmatrix} -1 & & & & \\ & & & \frac{1}{2\gamma} & \\ & & -1 & & \\ & 2\gamma & & & \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} & 1 & & \\ \gamma & & & \\ & & & 1 \\ & & \gamma & \end{pmatrix}$	$2q^2(q^4-1)$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5,\mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4,\mathbb{F}_q)$	**Order of centralizer**
B_{21}	$\begin{pmatrix} -1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& -1 \end{pmatrix}$	$\begin{pmatrix} 1 &&& \\ & 1 && \\ && -1 & \\ &&& -1 \end{pmatrix}$	$2q(q^2-1)(q-1)$
B_{22}	$\begin{pmatrix} 1 &&&& \\ &&& \frac{1}{2\gamma} & \\ && -1 && \\ & 2\gamma &&& \\ &&&& 1 \end{pmatrix}$	$\begin{pmatrix} & 1 && \\ \gamma &&& \\ &&& -1 \\ && -\gamma & \end{pmatrix}$	$2q(q^2-1)(q+1)$
B_3	$\begin{pmatrix} -1 &&& -1 & \\ & -1 &&& 1 \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{pmatrix}$	$\begin{pmatrix} -1 &&& 1 \\ & 1 && \\ && 1 & \\ &&& -1 \end{pmatrix}$	$q^2(q^2-1)$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5,\mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4,\mathbb{F}_q)$	**Order of centralizer**
B_{41}	$\begin{pmatrix} -1 & 2 & & -1 & -2 \\ & -1 & & & 1 \\ & & 1 & & \\ & & & -1 & -2 \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & & & 1 \\ & 1 & -1 & \\ & & 1 & \\ & & & -1 \end{pmatrix}$	$4q^2$
B_{42}	$\begin{pmatrix} -1 & 2\gamma & & -1 & -2\gamma \\ & -1 & & & 1 \\ & & 1 & & \\ & & & -1 & -2\gamma \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & & & 1 \\ & 1 & -\gamma & \\ & & 1 & \\ & & & -1 \end{pmatrix}$	
B_{43}	$\begin{pmatrix} -1 & & a & b & \frac{a^2}{2} \\ & & & \frac{1}{2\gamma} & \\ & & -1 & & -a \\ & 2\gamma & & & 2b\gamma \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} & 1 & & -\frac{a}{2} \\ \gamma & & -\frac{a\gamma}{2} & b\gamma \\ & & & 1 \\ & & \gamma & \end{pmatrix}$	
B_{44}	$\begin{pmatrix} -1 & -2c & & 2 & -4c \\ & & & \frac{1}{2\gamma} & -\frac{c}{\gamma} \\ & & -1 & & \\ & 2\gamma & & & 4\gamma \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} & 1 & c & \\ \gamma & & & 2\gamma \\ & & & 1 \\ & & \gamma & \end{pmatrix}$	
B_{51}	$\begin{pmatrix} -1 & & & & \\ & 1 & 1 & -\frac{1}{2} & \\ & & 1 & -1 & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & & -1 & -1 \\ & & & -1 \end{pmatrix}$	$2q(q-1)$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5, \mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4, \mathbb{F}_q)$	**Order of centralizer**
B_{52}	$\begin{pmatrix} 1 & \frac{1}{2\gamma} & & & \\ & & & 2\gamma & -1 \\ & & -1 & & \\ & \frac{1}{2\gamma} & & & \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} & 2\gamma & \frac{1}{2} & \\ \frac{1}{2} & & & \\ & & & -2\gamma \\ & & -\frac{1}{2} & \end{pmatrix}$	$2q(q+1)$
$C_1(i)$, $i \in T_1$	$\begin{pmatrix} \gamma^{-i} & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \gamma^i \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma^i & \\ & & & \gamma^i \end{pmatrix}$	$q(q^2-1)(q-1)$
$C_{21}(i)$, $i \in T_1$	$\begin{pmatrix} \gamma^{-i} & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \gamma^i \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \gamma^i & \\ & & & -\gamma^i \end{pmatrix}$	$2(q-1)^2$
$C_{22}(i)$, $i \in T_1$	$\begin{pmatrix} \gamma^{-i} & & & & \\ & & & \frac{1}{2\gamma} & \\ & & -1 & & \\ & 2\gamma & & & \\ & & & & \gamma^i \end{pmatrix}$	$\begin{pmatrix} & 1 & & \\ \gamma & & & \\ & & & -\gamma^i \\ & & -\gamma^{i+1} & \end{pmatrix}$	$2(q^2-1)$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5,\mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4,\mathbb{F}_q)$	**Order of centralizer**
$C_3(i)$, $i \in T_1$	$\begin{pmatrix} \gamma^{-i} & & & & \\ & 1 & 1 & -\frac{1}{2} & \\ & & 1 & -1 & \\ & & & 1 & \\ & & & & \gamma^{i} \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & & \gamma^{i} & \gamma^{i} \\ & & & \gamma^{i} \end{pmatrix}$	$q(q-1)$
$C_4(i)$ $i \in T_1$	$\begin{pmatrix} \gamma^{i} & & & \gamma^{i} & \\ & \gamma^{-i} & & & -\gamma^{-i} \\ & & 1 & & \\ & & & \gamma^{i} & \\ & & & & \gamma^{-i} \end{pmatrix}$	$\begin{pmatrix} 1 & & & -1 \\ & \gamma^{i} & & \\ & & \gamma^{-i} & \\ & & & 1 \end{pmatrix}$	$q(q-1)$
$C_5(i)$, $i \in T_1$	$\begin{pmatrix} \gamma^{i} & & & & \\ & \gamma^{i} & & & \\ & & 1 & & \\ & & & \gamma^{-i} & \\ & & & & \gamma^{-i} \end{pmatrix}$	$\begin{pmatrix} \gamma^{i} & & & \\ & 1 & & \\ & & 1 & \\ & & & \gamma^{-i} \end{pmatrix}$	$q(q^2-1)(q-1)$
$C_6(i,j)$, $i,j \in T_1$, $i < j$	$\begin{pmatrix} \gamma^{-j} & & & & \\ & \gamma^{-i} & & & \\ & & 1 & & \\ & & & \gamma^{i} & \\ & & & & \gamma^{j} \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & \gamma^{i} & & \\ & & \gamma^{j} & \\ & & & \gamma^{i+j} \end{pmatrix}$	$(q-1)^2$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5,\mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4,\mathbb{F}_q)$	**Order of centralizer**
$D_1(i)$, $i \in R_2$	$\begin{pmatrix} \theta^{-qi} & & & & \\ & \theta^{-i} & & & \\ & & 1 & & \\ & & & \theta^{i} & \\ & & & & \theta^{qi} \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & \theta^{i} & & \\ & & \theta^{qi} & \\ & & & \gamma^{i} \end{pmatrix}$	q^2-1
$D_2(i)$, $i \in T_2$	$\begin{pmatrix} 1 & & & & \\ & \eta^{i} & & & \\ & & 1 & & \\ & & & \eta^{-i} & \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} \theta^{i} & & & \\ & \theta^{qi} & & \\ & & \theta^{i} & \\ & & & \theta^{qi} \end{pmatrix}$	$q(q^2-1)(q+1)$
$D_{31}(i)$, $i \in T_2$	$\begin{pmatrix} -1 & & & & \\ & \eta^{i} & & & \\ & & 1 & & \\ & & & \eta^{-i} & \\ & & & & -1 \end{pmatrix}$	$\begin{pmatrix} \theta^{i} & & & \\ & \theta^{qi} & & \\ & & -\theta^{i} & \\ & & & -\theta^{qi} \end{pmatrix}$	$2(q^2-1)$
$D_{32}(i)$, $i \in T_2$	$\begin{pmatrix} & & & & 2 \\ & -\eta^{i} & & & \\ & & -1 & & \\ & & & -\eta^{-i} & \\ \frac{1}{2} & & & & \end{pmatrix}$	$D_{32}(i)$	$2(q+1)^2$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5, \mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4, \mathbb{F}_q)$	**Order of centralizer**
$D_4(i,j)$, $i \in T_2$, $j \in T_1$	$\begin{pmatrix} \gamma^{-j} & & & & \\ & \eta^i & & & \\ & & 1 & & \\ & & & \eta^{-i} & \\ & & & & \gamma^j \end{pmatrix}$	$\begin{pmatrix} \theta^{qi} & & & \\ & \theta^i & & \\ & & \gamma^j\theta^{qi} & \\ & & & \gamma^j\theta^i \end{pmatrix}$	$q^2 - 1$
$D_5(i)$, $i \in T_2$	$\begin{pmatrix} \eta^i & & & & \\ & 1 & 1 & -\frac{1}{2} & \\ & & 1 & -1 & \\ & & & 1 & \\ & & & & \eta^{-i} \end{pmatrix}$	$\begin{pmatrix} \theta^i & -\theta^i & & \\ & \theta^i & & \\ & & \theta^{qi} & \theta^{qi} \\ & & & \theta^{qi} \end{pmatrix}$	$q(q+1)$
$D_6(i)$, $i \in T_2$	$\begin{pmatrix} \eta^i & & & & \\ & \eta^{-i} & & & \\ & & 1 & & \\ & & & \eta^i & \\ & & & & \eta^{-i} \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & \eta^i & & \\ & & \eta^{-i} & \\ & & & 1 \end{pmatrix}$	$q(q^2-1)(q+1)$
$D_7(i,j)$, $i, j \in T_2$, $i < j$	$\begin{pmatrix} \eta^{-j} & & & & \\ & \eta^{-i} & & & \\ & & 1 & & \\ & & & \eta^i & \\ & & & & \eta^j \end{pmatrix}$	$D_7(i,j)$	$(q+1)^2$

Table 2.1 – Continued

Notation	**Class representative in** $\mathrm{SO}(5,\mathbb{F}_q)$	**Class representative in** $\mathrm{PGSp}(4,\mathbb{F}_q)$	**Order of centralizer**
$D_8(i)$ $i \in T_2$	$\begin{pmatrix} \eta^i & & & \eta^i & \\ & \eta^{-i} & & & -\eta^{-i} \\ & & 1 & & \\ & & & \eta^i & \\ & & & & \eta^{-i} \end{pmatrix}$	$\begin{pmatrix} 1 & & & -1 \\ & \eta^i & & \\ & & \eta^{-i} & \\ & & & 1 \end{pmatrix}$	$q(q+1)$
$D_9(i)$, $i \in R_1$	$\begin{pmatrix} \zeta^{-qi} & & & & \\ & \zeta^{-i} & & & \\ & & 1 & & \\ & & & \zeta^i & \\ & & & & \zeta^{qi} \end{pmatrix}$	$D_9(i)$	q^2+1

Explicit forms of the classes $D_{32}(i)$, $D_7(i,j)$, and $D_9(i)$ are omitted since they have a more complicated form and are not classes of the Borel, the Siegel parabolic, or the Klingen parabolic subgroup.

2.3 $\mathrm{GSp}(4,\mathbb{F}_q)$

The list of conjugacy classes of $\mathrm{PGSp}(4,\mathbb{F}_q)$ is used to determine the conjugacy classes of $\mathrm{GSp}(4,\mathbb{F}_q)$. Let's investigate how the class representatives of $\mathrm{PGSp}(4,\mathbb{F}_q)$ lead to representatives for the classes of $\mathrm{GSp}(4,\mathbb{F}_q)$.

Consider the natural projection map from $\mathrm{GSp}(4,\mathbb{F}_q)$ to $\mathrm{PGSp}(4,\mathbb{F}_q)$ given by

$$\mathrm{GSp}(4,\mathbb{F}_q) \longrightarrow \mathrm{PGSp}(4,\mathbb{F}_q),\ \ g \mapsto \overline{g}.$$

Let $g, h \in \mathrm{GSp}(4, \mathbb{F}_q)$. If $g = xhx^{-1}$, then, by taking multipliers on each side, it is clear that the multiplier of g is equal to the multiplier of h, i.e., $\lambda(g) = \lambda(h)$. Moreover, under the projection map,

$$\overline{g} = \overline{xhx^{-1}} = \overline{x} \cdot \overline{h} \cdot \overline{x^{-1}}.$$

So if two elements are conjugate in $\mathrm{GSp}(4, \mathbb{F}_q)$, they must be conjugate in $\mathrm{PGSp}(4, \mathbb{F}_q)$. The list of class representatives in $\mathrm{PGSp}(4, \mathbb{F}_q)$, when pulled back to $\mathrm{GSp}(4, \mathbb{F}_q)$, hit class representatives of all the conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$. Suppose now that two elements $g, h \in \mathrm{GSp}(4, \mathbb{F}_q)$ are conjugate in $\mathrm{PGSp}(4, \mathbb{F}_q)$, i.e. $\overline{g} = \overline{x} \cdot \overline{h} \cdot \overline{x^{-1}}$, for some $\overline{x} \in \mathrm{PGSp}(4, \mathbb{F}_q)$. Then, for some $\gamma^i \in \mathbb{F}_q^\times$,

$$g = \begin{pmatrix} \gamma^i & & & \\ & \gamma^i & & \\ & & \gamma^i & \\ & & & \gamma^i \end{pmatrix} xhx^{-1}.$$

Taking multipliers on both sides of the equation above, we have $\lambda(g) = \gamma^{2i}\lambda(h)$. So if the multiplier of g is a square, then the multiplier of h is a square and if the multiplier of g is a non-square, then the multiplier of h is a non-square. Write g and h in the following way

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma^{i_g} & \\ & & & \gamma^{i_g} \end{pmatrix} \cdot \begin{pmatrix} \gamma^{j_g} & & & \\ & \gamma^{j_g} & & \\ & & \gamma^{j_g} & \\ & & & \gamma^{j_g} \end{pmatrix} .g',$$

$$h = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma^{i_h} & \\ & & & \gamma^{i_h} \end{pmatrix} \cdot \begin{pmatrix} \gamma^{j_h} & & & \\ & \gamma^{j_h} & & \\ & & \gamma^{j_h} & \\ & & & \gamma^{j_h} \end{pmatrix} .h',$$

with $g', h' \in Sp(4, F_q)$, $i_g, i_h \in \{0,1\}$, and $j_g, j_h \in T_3$. So $\lambda(g) = \gamma^{i_g+2j_g}, \lambda(h) = \gamma^{i_h+2j_h}$. If g and h are conjugate, then $i_g = i_h$. Then $\gamma^{2j_h} = \gamma^{2j_h}$, or $(\gamma^{j_g})^2 = (\gamma^{j_h})^2$. So $\lambda(h) = \pm\lambda(g)$, i.e., the multipliers can only differ by a minus sign.

It is possible that an element g of $\mathrm{GSp}(4, \mathbb{F}_q)$ is conjugate to $-g$. An example is

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \cdot \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} & & -1 & \\ & & & -1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.$$

The centralizers are somewhat affected when we pull back our representatives in $\mathrm{PGSp}(4, \mathbb{F}_q)$ to $\mathrm{GSp}(4, \mathbb{F}_q)$. There are two types of pullbacks. The first type consists of elements g such that $g \neq x(-g)x^{-1}$ for any $x \in \mathrm{GSp}(4, \mathbb{F}_q)$. The second type consists of elements g such that $g = x(-g)x^{-1}$ for some $x \in \mathrm{GSp}(4, \mathbb{F}_q)$. Let $\mathrm{Cent}_{\mathrm{PGSp}}(\overline{g})$ denote the centralizer of $\overline{g}$ in $\mathrm{PGSp}(4, \mathbb{F}_q)$ and let $\mathrm{Cent}_{\mathrm{GSp}}(g)$ denote the centralizer of g in $\mathrm{GSp}(4, \mathbb{F}_q)$.

Type 1

Let $g \in \mathrm{GSp}(4, \mathbb{F}_q)$ be of the first type, i.e., g is not conjugate to $-g$. Let $\overline{h} \in \mathrm{Cent}_{\mathrm{PGSp}}(\overline{g})$. Then $\overline{g} = \overline{h} \cdot \overline{g} \cdot \overline{h^{-1}}$. When pulled back to $\mathrm{GSp}(4, \mathbb{F}_q)$, $g = z_0 hgh^{-1}$, with $z_0 = \pm I$. $z_0 \neq -I$ since g is not conjugate to $-g$. So $z_0 = I$, $g = hgh^{-1}$, and $h \in \mathrm{Cent}_{\mathrm{GSp}}(g)$. We get a short exact sequence

$$1 \longrightarrow Z \longrightarrow \mathrm{Cent}_{\mathrm{GSp}}(g) \longrightarrow \mathrm{Cent}_{\mathrm{PGSp}}(\overline{g}) \longrightarrow 1.$$

Therefore $\#\mathrm{Cent}_{\mathrm{GSp}}(g) = (q-1) \cdot \#\mathrm{Cent}_{\mathrm{PGSp}}(\overline{g})$.

Type 2

Let $g \in \mathrm{GSp}(4, \mathbb{F}_q)$ be of the second type, i.e., g is conjugate to $-g$. Define the set $S_g = \{h \in \mathrm{GSp}(4, F_q) : hgh^{-1} = -g\}$. Fix $s_0 \in S_g$. S_g is not a group, but there is a bijection of sets $S_g \longrightarrow \mathrm{Cent}_{\mathrm{GSp}}(g)$, given by the map $h \mapsto s_0 h$. Given $\overline{h} \in \mathrm{Cent}_{\mathrm{PGSp}}(\overline{g})$, either $h \in \mathrm{Cent}_{\mathrm{GSp}}(g)$ or $h \in S_g$. $S_g \sqcup \mathrm{Cent}_{\mathrm{GSp}}(g)$ maps onto $\mathrm{Cent}_{\mathrm{PGSp}}(\overline{g})$ via the projection map. Moreover, $\mathrm{Cent}'_{\mathrm{GSp}}(g) := S_g \sqcup \mathrm{Cent}_{\mathrm{GSp}}(g)$ is a group with respect to matrix multiplication and the projection map is a group homomorphism. $\mathrm{Cent}_{\mathrm{GSp}}(g)$ is a subgroup of $\mathrm{Cent}'_{\mathrm{GSp}}(g)$ of index 2. We get a short exact sequence

$$1 \longrightarrow Z \longrightarrow \mathrm{Cent}'_{\mathrm{GSp}}(g) \longrightarrow \mathrm{Cent}_{\mathrm{PGSp}}(\overline{g}) \longrightarrow 1.$$

Then $\#\mathrm{Cent}'_{\mathrm{GSp}}(g) = (q-1) \cdot \#\mathrm{Cent}_{\mathrm{PGSp}}(\overline{g})$. Also, $2 \cdot \#\mathrm{Cent}_{\mathrm{GSp}}(g) = \#\mathrm{Cent}'_{\mathrm{GSp}}(g)$. So $\mathrm{Cent}_{\mathrm{GSp}}(g) = \dfrac{q-1}{2} \cdot \#\mathrm{Cent}_{\mathrm{PGSp}}(\overline{g})$.

Thus, given a class representative $\overline{g} \in \mathrm{PGSp}(4, \mathbb{F}_q)$, we pull it back to $\mathrm{GSp}(4, \mathbb{F}_q)$. Then, we determine if g is of Type 1 or Type 2. If it is of Type 1, then there are $q-1$ conjugacy classes zg, for $z \in Z$, each of order $\dfrac{\#\mathrm{GSp}(4, \mathbb{F}_q)}{(q-1) \cdot \#\mathrm{Cent}_{\mathrm{PGSp}}(g)}$. If the pullback is of Type 2, then there are $\dfrac{q-1}{2}$ conjugacy classes $z_i g$, with

$$z_i = \begin{pmatrix} \gamma^i & & & \\ & \gamma^i & & \\ & & \gamma^i & \\ & & & \gamma^i \end{pmatrix}, \; i \in T_2.$$

Each of these classes is of order

$$\frac{\#\mathrm{GSp}(4, \mathbb{F}_q)}{\frac{(q-1)}{2} \cdot \#\mathrm{Cent}_{\mathrm{PGSp}}(g)}.$$

Explicitly, the list of conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$ is given in the following table.

Table 2.2: Conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$

Notation	**Class representative in** $\mathrm{GSp}(4, \mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$A_1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$\#\mathrm{GSp}(4, \mathbb{F}_q)$
$A_2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & \gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^4(q^2-1)(q-1)$
$A_{31}(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q-1)^2$
$A_{32}(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^{k+1} \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q^2-1)$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4,\mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$A_5(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & -\gamma^k & \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)$
$B_{11}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q^2(q^2-1)^2(q-1)$
$B_{12}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & & \\ \gamma^{k+1} & & & \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\frac{q-1}{2}$	$q^2(q^4-1)(q-1)$
$B_{21}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & -\gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q^2-1)(q-1)^2$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4,\mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$B_{22}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & & \\ \gamma^{k+1} & & & \\ & & & -\gamma^k \\ & & -\gamma^{k+1} & \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q^2-1)^2$
$B_3(k)$, $k \in T_3$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$q-1$	$q^2(q^2-1)(q-1)$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4, \mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$B_{41}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$,	$2q^2(q-1)$
$B_{42}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^{k+1} & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$,	
$B_{43}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & & -\frac{a}{2}\gamma^k \\ \gamma^{k+1} & & -\frac{a}{2}\gamma^{k+1} & b\gamma^{k+1} \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\frac{q-1}{2}$,	
$B_{44}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & c\gamma^k & \\ \gamma^{k+1} & & & 2\gamma^{k+1} \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\frac{q-1}{2}$	
$B_{51}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & -\gamma^k & -\gamma^k \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^2$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4,\mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$B_{52}(k)$, $k \in T_2$	$\begin{pmatrix} & 2\gamma^{k+1} & \frac{1}{2}\gamma^k & \\ \frac{1}{2}\gamma^k & & & \\ & & & -2\gamma^{k+1} \\ & & -\frac{1}{2}\gamma^k & \end{pmatrix}$	$\frac{q-1}{2}$	$q(q^2-1)$
$C_1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$
$C_{21}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^{k+i} & \\ & & & -\gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$C_{22}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} & \gamma^k & & \\ \gamma^{k+1} & & & \\ & & & -\gamma^{k+i} \\ & & -\gamma^{k+i+1} & \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q^2-1)(q-1)$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4,\mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$C_3(i,k)$, $i \in T_1$, $k \in T_3$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \gamma^{k+i} \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$C_4(i,k)$, $i \in T_1$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & -\gamma^k \\ & \gamma^{k+i} & & \\ & & \gamma^{k-i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$C_5(i,k)$, $i \in T_1$, $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^{k-i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$
$C_6(i,j,k)$, $i,j \in T_1$, $i < j$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k+j} & \\ & & & \gamma^{k+i+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4,\mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$D_1(i,k)$, $i \in R_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k\theta^i & & \\ & & \gamma^k\theta^{qi} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)^3}{4}$	$(q^2-1)(q-1)$
$D_2(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k\theta^i & & & \\ & \gamma^k\theta^{qi} & & \\ & & \gamma^k\theta^i & \\ & & & \gamma^k\theta^{qi} \end{pmatrix}$	$\frac{(q-1)^2}{2}$	$q(q^2-1)^2$
$D_{31}(i,k)$, $i,k \in T_2$	$\begin{pmatrix} \gamma^k\theta^i & & & \\ & \gamma^k\theta^{qi} & & \\ & & -\gamma^k\theta^i & \\ & & & -\gamma^k\theta^{qi} \end{pmatrix}$	$\frac{(q-1)^2}{4}$	$(q^2-1)(q-1)$
$D_{32}(i,k)$, $i,k \in T_2$	$D_{32}(i,k)$	$\frac{(q-1)^2}{4}$	$(q^2-1)(q+1)$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4,\mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$D_4(i,j,k)$, $i \in T_2$ $j \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k\theta^{qi} & & & \\ & \gamma^k\theta^i & & \\ & & \gamma^{k+j}\theta^{qi} & \\ & & & \gamma^{k+j}\theta^i \end{pmatrix}$	$\dfrac{(q-1)^2(q-3)}{4}$	$(q^2-1)(q-1)$
$D_5(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k\theta^i & -\gamma^k\theta^i & & \\ & \gamma^k\theta^i & & \\ & & \gamma^k\theta^{qi} & \gamma^k\theta^{qi} \\ & & & \gamma^k\theta^{qi} \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	$q(q^2-1)$
$D_6(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k\eta^i & & \\ & & \gamma^k\eta^{-i} & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	$q(q^2-1)^2$
$D_7(i,j,k)$, $i,j \in T_2$ $i < j$ $k \in T_3$	$D_7(i,j,k)$	$\dfrac{(q-1)^2(q-3)}{8}$	$(q^2-1)(q+1)$

Table 2.2 – Continued

Notation	**Class representative in** $\mathrm{GSp}(4,\mathbb{F}_q)$	**Number of classes**	**Order of centralizer**
$D_8(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & -\gamma^k \\ & \gamma^k\eta^i & & \\ & & \gamma^k\eta^{-i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)^2}{2}$	$q(q^2-1)$
$D_9(i,k)$, $i \in R_1$ $k \in T_3$	$D_9(i,k)$	$\frac{(q^2-1)(q-1)}{4}$	$(q^2+1)(q-1)$

There are $(q^2+2q+4)(q-1)$ conjugacy classes.

2.4 Borel, Siegel, and Klingen subgroups

The conjugacy classes of the Borel, the Siegel parabolic, and the Klingen parabolic subgroup can now be easily computed using Table 2.2. One does this by determining which conjugacy classes have a non-empty intersection with the subgroup, how each class splits, and computing the order of the centralizer of the class in the subgroup.

In each of the following tables of conjugacy classes, the notation will indicate which conjugacy classes of $\mathrm{GSp}(4,\mathbb{F}_q)$ occur in the subgroup and how many components the splitting has if the class splits into multiple classes in the subgroup. For example, the Borel subgroup is denoted by B. The class $A_2(k)$ has a non-empty intersection with B, splitting into two conjugacy classes, denoted by $BA_2^1(k)$ and $BA_2^2(k)$.

2.4.1 Borel

The Borel subgroup B of $\mathrm{GSp}(4,\mathbb{F}_q)$ is the set of all of the upper triangular matrices,

$$B = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{F}_q) \right\}.$$

Every element $g \in B$ can be written uniquely in the form

$$g = \begin{pmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

with $a, b, c \in \mathbb{F}_q^\times$ and $x, \kappa, \lambda, \mu \in \mathbb{F}_q$. The order of B is therefore $q^4(q-1)^3$. The multiplier of the matrix g given above is $\lambda(g) = c$. The subgroup of B of elements which have 1 on every entry on the main diagonal is $N_{\mathrm{GSp}(4)}$. The conjugacy classes of the Borel subgroup B are given in the following table.

Table 2.3: Conjugacy classes of the Borel subgroup

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BA_1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^4(q-1)^3$
$BA_2^1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & \gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^3(q-1)^2$
$BA_2^2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^k \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^4(q-1)^2$
$BA_{31}^1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q-1)$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BA_{31}^2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & \gamma^k & \\ & \gamma^k & & \gamma^k \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^3(q-1)^2$
$BA_{31}^3(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & & \\ & \gamma^k & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)^2$
$BA_{32}(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^{k+1} \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q-1)$
$BA_5(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & -\gamma^k & \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BB_{11}^{1}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q^2(q-1)^3$
$BB_{11}^{2}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q^2(q-1)^3$
$BB_{21}^{1}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & -\gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^3$
$BB_{21}^{2}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^3$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BB_{21}^3(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & \gamma^k & & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^3$
$BB_{21}^4(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^3$
$BB_3^1(k)$, $k \in T_3$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)^2$
$BB_3^2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & \gamma^k & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)^2$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BB_{41}^1(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$
$BB_{41}^2(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \frac{1}{4}\gamma^k \\ & -\gamma^k & -\gamma^k & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$
$BB_{42}^1(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^{k+1} & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$
$BB_{42}^2(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \frac{1}{4}\gamma^{k+1} \\ & -\gamma^k & -\gamma^k & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BB_{51}^{1}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & -\gamma^k & -\gamma^k \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$
$BB_{51}^{2}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & -\gamma^k & & \\ & -\gamma^k & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$
$BB_{51}^{3}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & -\gamma^k & & \\ & \gamma^k & \gamma^k & -\gamma^k \\ & & -\gamma^k & \gamma^k \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$
$BB_{51}^{4}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & \gamma^k & & \\ & -\gamma^k & -\gamma^k & \gamma^k \\ & & \gamma^k & -\gamma^k \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_1^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$BC_1^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$BC_1^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$BC_1^4(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_{21}^{1}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^{k+i} & \\ & & & -\gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$BC_{21}^{2}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & -\gamma^{k+i} & & \\ & & \gamma^{k} & \\ & & & -\gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$BC_{21}^{3}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k} & & \\ & & -\gamma^{k+i} & \\ & & & -\gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$BC_{21}^{4}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} -\gamma^{k+i} & & & \\ & \gamma^{k+i} & & \\ & & -\gamma^{k} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_{21}^{5}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} -\gamma^{k+i} & & & \\ & -\gamma^{k} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$BC_{21}^{6}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^{k+i} & & \\ & & -\gamma^{k} & \\ & & & -\gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$BC_{21}^{7}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} -\gamma^{k} & & & \\ & -\gamma^{k+i} & & \\ & & \gamma^{k} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$BC_{21}^{8}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} -\gamma^{k} & & & \\ & \gamma^{k} & & \\ & & -\gamma^{k+i} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_3^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \gamma^{k+i} \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$BC_3^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & \gamma^{k+i} & & \\ & \gamma^{k+i} & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$BC_3^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & \gamma^{k+i} & \\ & \gamma^k & & \gamma^k \\ & & \gamma^{k+i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$BC_3^4(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & \gamma^k & \\ & \gamma^{k+i} & & \gamma^{k+i} \\ & & \gamma^k & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_4^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & -\gamma^k \\ & \gamma^{k+i} & & \\ & & \gamma^{k-i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$BC_4^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & \gamma^k & \\ & \gamma^{k-i} & & \gamma^{k-i} \\ & & \gamma^{k+i} & -\gamma^{k+i} \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$BC_4^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k-i} & & & \\ & \gamma^k & \gamma^k & \\ & & \gamma^k & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$BC_4^4(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k & \gamma^k & \\ & & \gamma^k & \\ & & & \gamma^{k-i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_5^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^{k-i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$BC_5^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k-i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$BC_5^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^{k-i} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$BC_5^4(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k-i} & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_6^1(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k+j} & \\ & & & \gamma^{k+i+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$BC_6^2(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i+j} & & & \\ & \gamma^{k+j} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$BC_6^3(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i+j} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k+j} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$BC_6^4(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+j} & & & \\ & \gamma^{k} & & \\ & & \gamma^{k+i+j} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in B**	**Order of centralizer in B**
$BC_6^5(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+j} & & & \\ & \gamma^{k+i+j} & & \\ & & \gamma^{k} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$BC_6^6(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k+i+j} & & \\ & & \gamma^{k} & \\ & & & \gamma^{k+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$BC_6^7(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k} & & \\ & & \gamma^{k+i+j} & \\ & & & \gamma^{k+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$

Table 2.3 – Continued

Notation	**Class representative**	**Number of classes in** B	**Order of centralizer in** B
$BC_6^8(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^k &&& \\ & \gamma^{k+j} && \\ && \gamma^{k+i} & \\ &&& \gamma^{k+i+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$

2.4.2 Siegel

The Siegel parabolic subgroup P of $\mathrm{GSp}(4,\mathbb{F}_q)$ is defined as

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ && * & * \\ && * & * \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{F}_q) \right\}.$$

Every element $p \in P$ can be written uniquely in the form

$$p = \begin{pmatrix} a & b && \\ c & d && \\ && \lambda a/\Delta & -\lambda b/\Delta \\ && -\lambda c/\Delta & \lambda d/\Delta \end{pmatrix} \cdot \begin{pmatrix} 1 && \mu & \kappa \\ & 1 & x & \mu \\ && 1 & \\ &&& 1 \end{pmatrix},$$

with $\Delta = ad - bc \in \mathbb{F}_q^\times$, $\lambda \in \mathbb{F}_q^\times$ and $x, \kappa, \mu \in \mathbb{F}_q$. The order of P is therefore $q^4(q^2-1)(q-1)^2$. The multiplier of p is $\lambda(p) = \lambda$. We also define

$$A' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^tA^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

for any $A \in \mathrm{GL}(2, \mathbb{F}_q)$.

The conjugacy classes of the Siegel parabolic subgroup P are given in the following table.

Table 2.4: Conjugacy classes of the Siegel parabolic subgroup

Notation	**Class representative**	**Number of classes in** P	**Order of centralizer in** P
$PA_1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k &&& \\ & \gamma^k && \\ && \gamma^k & \\ &&& \gamma^k \end{pmatrix}$	$q-1$	$q^4(q^2-1)(q-1)^2$
$PA_2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k &&& \\ & \gamma^k & \gamma^k & \\ && \gamma^k & \\ &&& \gamma^k \end{pmatrix}$	$q-1$	$q^4(q-1)^2$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PA_{31}^{1}(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q-1)^2$
$PA_{31}^{2}(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & & \\ & \gamma^k & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)^2$
$PA_{32}(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^{k+1} \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q^2-1)$
$PA_5(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & -\gamma^k & \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PB_{11}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q^2(q-1)^3$
$PB_{12}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & & \\ \gamma^{k+1} & & & \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\frac{q-1}{2}$	$q^2(q^2-1)(q-1)$
$PB_{21}^1(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & -\gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q^2-1)(q-1)^2$
$PB_{21}^2(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q^2-1)(q-1)^2$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PB_{21}^3(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & \gamma^k & & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^3$
$PB_{22}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & & \\ \gamma^{k+1} & & & \\ & & & -\gamma^k \\ & & -\gamma^{k+1} & \end{pmatrix}$	$\frac{q-1}{2}$	$q(q^2-1)(q-1)$
$PB_3(k)$, $k \in T_3$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)^2$
$PB_{41}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PB_{42}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^{k+1} & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$
$PB_{43}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & & -\frac{a\gamma^k}{2} \\ \gamma^{k+1} & & -\frac{a\gamma^{k+1}}{2} & b\gamma^{k+1} \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$
$PB_{44}(k)$, $k \in T_2$	$\begin{pmatrix} & \gamma^k & c\gamma^k & \\ \gamma^{k+1} & & & 2\gamma^{k+1} \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\frac{q-1}{2}$	$2q^2(q-1)$
$PB_{51}^1(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & -\gamma^k & -\gamma^k \\ & & & -\gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^2$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PB_{51}^{2}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & -\gamma^k & & \\ & -\gamma^k & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$
$PB_{51}^{3}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & -\gamma^k & \\ \gamma^k & -\gamma^k & -\gamma^{-k} & \gamma^k \\ & & \gamma^k & \\ & & -\gamma^k & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$
$PB_{52}(k)$, $k \in T_2$	$\begin{pmatrix} & 2\gamma^{k+1} & \frac{1}{2}\gamma^k & \\ \frac{1}{2}\gamma^k & & & \\ & & & -2\gamma^{k+1} \\ & & -\frac{1}{2}\gamma^k & \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q^2-1)$
$PC_1^1(i,k)$, $i \in T_1$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PC_1^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$
$PC_1^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$PC_{21}^1(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k} & & & \\ & -\gamma^{k} & & \\ & & \gamma^{k+i} & \\ & & & -\gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$PC_{21}^2(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & -\gamma^{k+i} & & \\ & & \gamma^{k} & \\ & & & -\gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PC_{21}^3(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k & & \\ & & -\gamma^{k+i} & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$PC_{21}^4(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} -\gamma^{k+i} & & & \\ & -\gamma^k & & \\ & & \gamma^{k+i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$PC_{22}^1(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} & \gamma^k & & \\ \gamma^{k+1} & & & \\ & & & -\gamma^{k+i} \\ & & -\gamma^{k+i+1} & \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q^2-1)(q-1)$
$PC_{22}^2(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} & -\gamma^{k+i} & & \\ -\gamma^{k+i+1} & & & \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q^2-1)(q-1)$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PC_3^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \gamma^{k+i} \\ & & & \gamma^{k+i} \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$PC_3^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & \gamma^{k+i} & & \\ & \gamma^{k+i} & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$PC_3^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & \gamma^{k+i} & \\ & \gamma^k & & \gamma^k \\ & & \gamma^{k+i} & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$PC_4^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & -\gamma^k \\ & \gamma^{k+i} & & \\ & & \gamma^{k-i} & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q-1)^2$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PC_4^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & \gamma^k & \\ & \gamma^{k-i} & & \gamma^{k-i} \\ & & \gamma^{k+i} & \gamma^{k+i} \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$PC_5^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^{k-i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$PC_5^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^{k-i} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$PC_6^1(i,j,k)$, $i,j \in T_1$ $i < j$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k+j} & \\ & & & \gamma^{k+i+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PC_6^2(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i+j} & & & \\ & \gamma^{k+j} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$PC_6^3(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i+j} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k+j} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$PC_6^4(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+j} & & & \\ & \gamma^{k} & & \\ & & \gamma^{k+i+j} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PD_2(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k\theta^i & & & \\ & \gamma^k\theta^{qi} & & \\ & & \gamma^k\theta^i & \\ & & & \gamma^k\theta^{qi} \end{pmatrix}$	$\frac{(q-1)^2}{2}$	$q(q^2-1)(q-1)$
$PD_{31}^1(i,k)$, $i,k \in T_2$	$\begin{pmatrix} \gamma^k\theta^i & & & \\ & \gamma^k\theta^{qi} & & \\ & & -\gamma^k\theta^i & \\ & & & -\gamma^k\theta^{qi} \end{pmatrix}$	$\frac{(q-1)^2}{4}$	$(q^2-1)(q-1)$
$PD_{31}^2(i,k)$, $i,k \in T_2$	$\begin{pmatrix} -\gamma^k\theta^i & & & \\ & -\gamma^k\theta^{qi} & & \\ & & \gamma^k\theta^i & \\ & & & \gamma^k\theta^{qi} \end{pmatrix}$	$\frac{(q-1)^2}{4}$	$(q^2-1)(q-1)$
$PD_4^1(i,j,k)$, $i \in T_2$ $j \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k\theta^{qi} & & & \\ & \gamma^k\theta^i & & \\ & & \gamma^{k+j}\theta^{qi} & \\ & & & \gamma^{k+j}\theta^i \end{pmatrix}$	$\frac{(q-1)^2(q-3)}{4}$	$(q^2-1)(q-1)$

Table 2.4 – Continued

Notation	**Class representative**	**Number of classes in P**	**Order of centralizer in P**
$PD_4^2(i,j,k)$, $i \in T_2$ $j \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+j}\theta^{qi} & & & \\ & \gamma^{k+j}\theta^{i} & & \\ & & \gamma^{k}\theta^{qi} & \\ & & & \gamma^{k}\theta^{i} \end{pmatrix}$	$\dfrac{(q-1)^2(q-3)}{4}$	$(q^2-1)(q-1)$
$PD_5(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k\theta^i & -\gamma^k\theta^i & & \\ & \gamma^k\theta^i & & \\ & & \gamma^k\theta^{qi} & \gamma^k\theta^{qi} \\ & & & \gamma^k\theta^{qi} \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	$q(q^2-1)$

2.4.3 Klingen

The Klingen parabolic subgroup Q of $\mathrm{GSp}(4,\mathbb{F}_q)$ is defined as

$$Q = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{F}_q) \right\}.$$

Every element $g \in Q$ can be written uniquely in the form

$$g = \begin{pmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \Delta t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

with $\Delta = ad - bc \in \mathbb{F}_q^\times$, $t \in \mathbb{F}_q^\times$, and $\kappa, \lambda, \mu \in \mathbb{F}_q$. The order of Q is therefore $q^4(q^2-1)(q-1)^2$. The multiplier of g given above is $\lambda(g) = \Delta$.

The conjugacy classes of the Klingen parabolic subgroup Q are given in the following table.

Table 2.5: Conjugacy classes of the Klingen parabolic subgroup

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QA_1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^4(q^2-1)(q-1)^2$
$QA_2^1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & \gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^3(q-1)^2$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QA_2^2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^k \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^4(q^2-1)(q-1)$
$QA_{31}^1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q-1)$
$QA_{31}^2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & \gamma^k & \\ & \gamma^k & & \gamma^k \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^3(q-1)^2$
$QA_{32}(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^{k+1} \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q-1)$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QA_5(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & -\gamma^k & \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)$
$QB^1_{11}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q^2(q^2-1)(q-1)^2$
$QB^2_{11}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q^2(q^2-1)(q-1)^2$
$QB^1_{21}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & -\gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^3$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QB_{21}^2(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^3$
$QB_3^1(k)$, $k \in T_3$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$q-1$	$q^2(q^2-1)(q-1)$
$QB_3^2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & \gamma^k & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)^2$
$QB_{41}^1(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$2q^2(q-1)$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QB_{41}^{2}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \frac{1}{4}\gamma^k \\ & -\gamma^k & -\gamma^k & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$2q^2(q-1)$
$QB_{42}^{1}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^{k+1} & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$2q^2(q-1)$
$QB_{42}^{2}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \frac{1}{4}\gamma^{k+1} \\ & -\gamma^k & -\gamma^k & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$2q^2(q-1)$
$QB_{51}^{1}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & -\gamma^k & -\gamma^k \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QB_{51}^2(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & -\gamma^k & & \\ & -\gamma^k & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$\frac{q-1}{2}$	$q(q-1)^2$
$QC_1^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$QC_1^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$QC_{21}^1(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^{k+i} & \\ & & & -\gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QC_{21}^2(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & -\gamma^{k+i} & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$QC_{21}^3(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} -\gamma^{k+i} & & & \\ & \gamma^{k+i} & & \\ & & -\gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$QC_{21}^4(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} -\gamma^{k} & & & \\ & -\gamma^{k+i} & & \\ & & \gamma^k & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{4}$	$(q-1)^3$
$QC_3^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & -\gamma^{k} & & \\ & \gamma^{k} & & \\ & & \gamma^{k+i} & \gamma^{k+i} \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QC_3^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & \gamma^{k+i} & & \\ & \gamma^{k+i} & & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$QC_4^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & -\gamma^k \\ & \gamma^{k+i} & & \\ & & \gamma^{k-i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$QC_4^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k-i} & & & \\ & \gamma^{k} & \gamma^k & \\ & & \gamma^{k} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$QC_4^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k} & \gamma^k & \\ & & \gamma^{k} & \\ & & & \gamma^{k-i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^2$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QC_5^1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^{k-i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$
$QC_5^2(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k-i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q-1)^3$
$QC_5^3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k-i} & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$
$QC_6^1(i,j,k)$, $i,j \in T_1$ $i < j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k+j} & \\ & & & \gamma^{k+i+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QC_6^2(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i+j} & & & \\ & \gamma^{k+j} & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$QC_6^3(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+j} & & & \\ & \gamma^{k} & & \\ & & \gamma^{k+i+j} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$QC_6^4(i,j,k)$, $i,j \in T_1$ $i<j$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^{k+i+j} & & \\ & & \gamma^{k} & \\ & & & \gamma^{k+j} \end{pmatrix}$	$\frac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$QD_1^1(i,k)$, $i \in R_2$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^{k}\theta^{i} & & \\ & & \gamma^{k}\theta^{qi} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\frac{(q-1)^3}{4}$	$(q^2-1)(q-1)$

Table 2.5 – Continued

Notation	**Class representative**	**Number of classes in Q**	**Order of centralizer in Q**
$QD_1^2(i,k)$, $i \in R_2$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k\theta^i & & \\ & & \gamma^k\theta^{qi} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)^3}{4}$	$(q^2-1)(q-1)$
$QD_6(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & \\ & \gamma^k\eta^i & & \\ & & \gamma^k\eta^{-i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)^2}{2}$	$q(q^2-1)(q-1)$
$QD_8(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^{k} & & & -\gamma^k \\ & \gamma^k\eta^i & & \\ & & \gamma^k\eta^{-i} & \\ & & & \gamma^k \end{pmatrix}$	$\frac{(q-1)^2}{2}$	$q(q^2-1)$

2.5 $N_{\mathrm{GSp}(4)}$

Recall that the subgroup $N_{\mathrm{GSp}(4)}$ of $\mathrm{GSp}(4,\mathbb{F}_q)$ is defined as

$$N_{\mathrm{GSp}(4)} = \left\{ \begin{pmatrix} 1 & y & * & * \\ & 1 & x & * \\ & & 1 & -y \\ & & & 1 \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{F}_q) \right\}.$$

Every element $g \in N_{\mathrm{GSp}(4)}$ can be written uniquely in the form

$$g = \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

with $x, \kappa, \lambda, \mu \in \mathbb{F}_q$. The order of N is therefore q^4. The multiplier of the matrix g given above is $\lambda(g) = 1$. The conjugacy classes of $N = N_{\mathrm{GSp}(4)}$ are listed in the following table. Note that the notation is slightly different than the standard notation we have been using for conjugacy classes of subgroups of $\mathrm{GSp}(4, \mathbb{F}_q)$.

Table 2.6: Conjugacy classes of $N_{\mathrm{GSp}(4)}$

Notation	**Class representative**	**Number of classes in** N	**Order of centralizer in** N
NA_1	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	1	q^4
$NA_2^1(k)$, $k \in T_3$	$\begin{pmatrix} 1 & & & \\ & 1 & \gamma^k & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$q-1$	q^3

Table 2.6 – Continued

Notation	**Class Representative**	**Number of Classes in N**	**Order of Centralizer in N**
$NA_2^2(k)$, $k \in T_3$	$\begin{pmatrix} 1 & & & \gamma^k \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$q-1$	q^4
$NA_{31}^1(i,j,\kappa)$, $i,j \in T_3$ $\gamma^{2i} - \gamma^j\kappa = \gamma^{2n}$ for some $n \in T_3$	$\begin{pmatrix} 1 & & \gamma^i & \kappa \\ & 1 & \gamma^j & \gamma^i \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	q^3
$NA_{31}^2(k)$, $k \in T_3$	$\begin{pmatrix} 1 & \gamma^k & & \\ & 1 & & \\ & & 1 & -\gamma^k \\ & & & 1 \end{pmatrix}$	$q-1$	q^2
$NA_{31}^3(K)$, $k \in T_3$	$\begin{pmatrix} 1 & & \gamma^k & \\ & 1 & & \gamma^k \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$q-1$	q^3

Table 2.6 – Continued

Notation	**Class Representative**	**Number of Classes in N**	**Order of Centralizer in N**
$NA_{32}(i,j,\kappa)$, $i,j \in T_3$ $\gamma^{2i}-\gamma^j\kappa=\gamma^{2n+1}$ for some $n \in T_3$	$\begin{pmatrix} 1 & & \gamma^i & \kappa \\ & 1 & \gamma^j & \gamma^i \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	q^3
$NA_5(i,j)$, $i,j \in T_3$	$\begin{pmatrix} 1 & \gamma^i & & \\ & 1 & \gamma^j & -\gamma^{i+j} \\ & & 1 & -\gamma^i \\ & & & 1 \end{pmatrix}$	$(q-1)^2$	q^2

Chapter 3

Induced characters

Let C be a conjugacy class of a group G. Then the conjugacy class C splits into distinct conjugacy classes of a subgroup H, say $C = D_1 \sqcup \ldots \sqcup D_r$. The value of the induced character is given by the formula

$$\mathrm{Ind}_H^G(\chi)(C) = \chi^G(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi(D_i).$$

This formula is used to find the induced character values of representations defined on the Borel, the Siegel parabolic, and the Klingen parabolic subgroups. This will lead to the complete list of the irreducible non–cuspidal characters of $\mathrm{GSp}(4, \mathbb{F}_q)$. This formula is also used to find the character values of the representation $\mathcal{G}$ to determine the genericity of characters. This chapter includes the character tables of representations induced from B, P, and Q as well as the character table of $\mathcal{G}$. If a conjugacy class of $\mathrm{GSp}(4, \mathbb{F}_q)$ is not listed in the table, the character takes the value 0 on that conjugacy class.

3.1 Parabolic induction

The conjugacy classes of $\mathrm{GSp}(4,\mathbb{F}_q)$,B, P, and Q have been determined so one can easily find all of the parabolically induced character values.

3.1.1 Borel

Let χ_1, χ_2, and σ be characters of the multiplicative group $\mathbb{F}_q^\times$. Define a character on the Borel subgroup B by

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} \longmapsto \chi_1(a)\chi_2(b)\sigma(c).$$

The character of this representation is given by $\chi_1(a)\chi_2(b)\sigma(c)$. This representation is induced to obtain a representation of $\mathrm{GSp}(4,\mathbb{F}_q)$, denoted by $\chi_1 \times \chi_2 \rtimes \sigma$. The standard model of this representation is the space of functions

$$f : \mathrm{GSp}(4,\mathbb{F}_q) \longrightarrow \mathbb{C}$$

satisfying

$$f(hg) = \chi_1(a)\chi_2(b)\sigma(c)f(g), \quad \text{for all } h = \begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} \in B.$$

The group action is by right translation. The central character of $\chi_1 \times \chi_2 \rtimes \sigma$ is $\chi_1\chi_2\sigma^2$. The character table of $\chi_1 \times \chi_2 \rtimes \sigma$ is the following.

Table 3.1: $\chi_1 \times \chi_2 \rtimes \sigma$ character values

Conjugacy class	$\chi_1 \times \chi_2 \rtimes \sigma$ **character value**
$A_1(k)$	$(q^2+1)(q+1)^2\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_2(k)$	$(q+1)^2\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_{31}(k)$	$(3q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_{32}(k)$	$(q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_5(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$B_{11}(k)$	$(q+1)^2\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1)+\chi_2(-1)\Big)$
$B_{21}(k)$	$(q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(-\gamma^{2k})\Big(1+\chi_1(-1)\chi_2(-1)+\chi_1(-1)+\chi_2(-1)\Big)$
$B_3(k)$	$(q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1)+\chi_2(-1)\Big)$
$B_{41}(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1)+\chi_2(-1)\Big)$
$B_{42}(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1)+\chi_2(-1)\Big)$
$B_{51}(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(-\gamma^{2k})\Big(1+\chi_1(-1)\chi_2(-1)+\chi_1(-1)+\chi_2(-1)\Big)$
$C_1(i,k)$	$(q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k+i})\Big(1+\chi_1(\gamma^i)\chi_2(\gamma^i)+\chi_1(\gamma^i)+\chi_2(\gamma^i)\Big)$

Table 3.1 – Continued

Conjugacy class	$\chi_1 \times \chi_2 \rtimes \sigma$ **character value**
$C_{21}(i,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(-\gamma^{2k+i})\Big(\chi_1(-1)+\chi_2(-1)$ $+\chi_1(-1)\chi_2(-1)(\chi_1(\gamma^i)+\chi_2(\gamma^i))$ $+(\chi_1(-1)+\chi_2(-1))\chi_1(\gamma^i)\chi_2(\gamma^i)+\chi_1(\gamma^i)+\chi_2(\gamma^i)\Big)$
$C_3(i,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k+i})\Big(1+\chi_1(\gamma^i)\chi_2(\gamma^i)+\chi_1(\gamma^i)+\chi_2(\gamma^i)\Big)$
$C_4(i,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(\gamma^i)+\chi_1(\gamma^{-i})+\chi_2(\gamma^i)+\chi_2(\gamma^{-i})\Big)$
$C_5(i,k)$	$(q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(\gamma^i)+\chi_1(\gamma^{-i})+\chi_2(\gamma^i)+\chi_2(\gamma^{-i})\Big)$
$C_6(i,j,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k+i+j})\Big(\chi_1(\gamma^i)+\chi_1(\gamma^j)+\chi_2(\gamma^i)+\chi_2(\gamma^j)$ $+\chi_1(\gamma^{i+j})(\chi_2(\gamma^i)+\chi_2(\gamma^j))+\chi_2(\gamma^{i+j})\big(\chi_1(\gamma^i)+\chi_1(\gamma^j)\big)\Big)$

3.1.2 Siegel

Let (π, V) be an irreducible representation of $\mathrm{GL}(2,\mathbb{F}_q)$, and let σ be a character of $\mathbb{F}_q^\times$. Define a representation of the Siegel parabolic subgroup P on V by

$$\begin{pmatrix} A & * \\ & cA' \end{pmatrix} \longmapsto \sigma(c)\pi(A).$$

The character of this representation is given by $\sigma(c)\chi_\pi(A)$, where χ_π is the character of π. This representation is induced to obtain a representation of $\mathrm{GSp}(4,\mathbb{F}_q)$, denoted by $\pi \rtimes \sigma$. The standard model of this representation is the space of functions

$$f : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow V$$

satisfying

$$f(hg) = \sigma(c)\pi(A)f(g), \quad \text{for all } h = \begin{pmatrix} A & * \\ & cA' \end{pmatrix} \in P.$$

The group action is by right translation. If π has central character ω_π, then the central character of $\pi \rtimes \sigma$ is $\omega_\pi \sigma^2$. The character table of $\pi \rtimes \sigma$ is the following.

Table 3.2: $\pi \rtimes \sigma$ character values

Conjugacy class	$\pi \rtimes \sigma$ **character value**
$A_1(k)$	$(q^2+1)(q+1)\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$
$A_2(k)$	$(q+1)\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$
$A_{31}(k)$	$\sigma(\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix}) + 2q\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})\Big)$
$A_{32}(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$
$A_5(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})$
$B_{11}(k)$	$(q+1)^2\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})$

Table 3.2 – Continued

Conjugacy class	$\pi \rtimes \sigma$ **character value**
$B_{12}(k)$	$(q^2+1)\sigma(\gamma^{2k+1})\chi_\pi(\begin{pmatrix}\gamma^{k+1/2} & \\ & -\gamma^{k+1/2}\end{pmatrix})$
$B_{21}(k)$	$\sigma(-\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & \gamma^{k}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^{k} & \\ & -\gamma^{k}\end{pmatrix})$ $+ (q+1)\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & -\gamma^{k}\end{pmatrix})\Big)$
$B_{22}(k)$	$(q+1)\sigma(-\gamma^{2k+1})\chi_\pi(\begin{pmatrix}\gamma^{k+1/2} & \\ & -\gamma^{k+1/2}\end{pmatrix})$
$B_{3}(k)$	$(q+1)\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & -\gamma^{k}\end{pmatrix})$
$B_{41}(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & -\gamma^{k}\end{pmatrix})$
$B_{42}(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & -\gamma^{k}\end{pmatrix})$
$B_{43}(k)$	$\sigma(\gamma^{2k+1})\chi_\pi(\begin{pmatrix}\gamma^{k+1/2} & \\ & -\gamma^{k+1/2}\end{pmatrix})$
$B_{44}(k)$	$\sigma(\gamma^{2k+1})\chi_\pi(\begin{pmatrix}\gamma^{k+1/2} & \\ & -\gamma^{k+1/2}\end{pmatrix})$
$B_{51}(k)$	$\sigma(-\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & -\gamma^{k}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k} & 1\\ & \gamma^{k}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^{k} & 1\\ & -\gamma^{k}\end{pmatrix})\Big)$
$B_{52}(k)$	$\sigma(-\gamma^{2k+1})\chi_\pi(\begin{pmatrix}\gamma^{k+1/2} & \\ & -\gamma^{k+1/2}\end{pmatrix})$
$C_1(i,k)$	$\sigma(\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & \gamma^{k}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k+i}\end{pmatrix})$ $+(q+1)\chi_\pi(\begin{pmatrix}\gamma^{k} & \\ & \gamma^{k+i}\end{pmatrix})\Big)$

Table 3.2 – Continued

Conjugacy class	$\pi \rtimes \sigma$ **character value**
$C_{21}(i,k)$	$\sigma(-\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix}\gamma^k & \\ & -\gamma^k\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & -\gamma^{k+i}\end{pmatrix})$ $+\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^k & \\ & -\gamma^{k+i}\end{pmatrix})\Big)$
$C_{22}(i,k)$	$\sigma(-\gamma^{2k+i+1})\Big(\chi_\pi(\begin{pmatrix}\gamma^{k+1/2} & \\ & -\gamma^{k+1/2}\end{pmatrix})$ $+ \chi_\pi(\begin{pmatrix}\gamma^{k+i+1/2} & \\ & -\gamma^{k+i+1/2}\end{pmatrix})\Big)$
$C_3(i,k)$	$\sigma(\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+i} & 1\\ & \gamma^{k+i}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix})\Big)$
$C_4(i,k)$	$\sigma(\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k-i}\end{pmatrix})\Big)$
$C_5(i,k)$	$(q+1)\sigma(\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k-i}\end{pmatrix})\Big)$
$C_6(i,j,k)$	$\sigma(\gamma^{2k+i+j})\Big(\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+j}\end{pmatrix})$ $+\chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k+i+j}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+j} & \\ & \gamma^{k+i+j}\end{pmatrix})\Big)$
$D_2(i,k)$	$(q+1)\sigma(\gamma^{2k+i})\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix})$
$D_{31}(i,k)$	$\sigma(-\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^k\theta^i & \\ & -\gamma^k\theta^{qi}\end{pmatrix})\Big)$
$D_4(i,j,k)$	$\sigma(\gamma^{2k+i+j})\Big(\chi_\pi(\begin{pmatrix}\gamma^k\theta^{qi} & \\ & \gamma^k\theta^i\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+j}\theta^{qi} & \\ & \gamma^{k+j}\theta^i\end{pmatrix}\Big)$
$D_5(i,k)$	$\sigma(\gamma^{2k+i})\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix})$

3.1.3 Klingen

Let χ be a character of $\mathbb{F}_q^\times$, and let (π, V) be an irreducible representation of $\mathrm{GL}(2, \mathbb{F}_q)$. Define a representation of the Klingen parabolic subgroup Q on V by

$$\begin{pmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{pmatrix} \longmapsto \chi(t)\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}), \quad \text{where } \Delta = ad - bc.$$

The character of this representation is given by $\chi(t)\chi_\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$, where χ_π is the character of π. This representation is induced to obtain a representation of $\mathrm{GSp}(4, \mathbb{F}_q)$, denoted by $\chi \rtimes \pi$. The standard model of this representation is the space of functions

$$f : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow V$$

satisfying

$$f(hg) = \chi(t)\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})f(g), \quad \text{for all } h = \begin{pmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{pmatrix} \in Q.$$

The group action is by right translation. If π has central character ω_π, then the central character of $\chi \rtimes \pi$ is $\chi\omega_\pi$. The character table of $\chi \rtimes \pi$ is the following.

Table 3.3: $\chi \rtimes \pi$ character values

Conjugacy class	$\chi \rtimes \pi$ **character value**
$A_1(k)$	$(q^2+1)(q+1)\chi(\gamma^k)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$

Table 3.3 – Continued

Conjugacy class	$\chi \rtimes \pi$ **character value**
$A_2(k)$	$\chi(\gamma^k)\Big(\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^k\end{pmatrix}) + q(q+1)\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix})\Big)$
$A_{31}(k)$	$2\chi(\gamma^k)\Big(\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^k\end{pmatrix}) + \frac{(q-1)}{2}\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix})\Big)$
$A_{32}(k)$	$(q+1)\chi(\gamma^k)\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix})$
$A_5(k)$	$\chi(\gamma^k)\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix})$
$B_{11}(k)$	$(q+1)\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^k\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^k & \\ & -\gamma^k\end{pmatrix})\Big)$
$B_{21}(k)$	$(q+1)\chi(\gamma^k)\Big(1+\chi(-1)\Big)\chi_\pi(\begin{pmatrix}\gamma^k & \\ & -\gamma^k\end{pmatrix})$
$B_3(k)$	$\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^k\end{pmatrix}) + (q+1)\chi_\pi(\begin{pmatrix}-\gamma^k & 1\\ & -\gamma^k\end{pmatrix})\Big)$
$B_{41}(k)$	$\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^k & 1\\ & -\gamma^k\end{pmatrix})\Big)$
$B_{42}(k)$	$\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^k & 1\\ & -\gamma^k\end{pmatrix})\Big)$
$B_{51}(k)$	$\chi(\gamma^k)\Big(1+\chi(-1)\Big)\chi_\pi(\begin{pmatrix}\gamma^k & \\ & -\gamma^k\end{pmatrix})$
$C_1(i,k)$	$(q+1)\chi(\gamma^k)\Big(1+\chi(\gamma^i)\Big)\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix})$

Table 3.3 – Continued

Conjugacy class	$\chi \rtimes \pi$ **character value**
$C_{21}(i,k)$	$\chi(\gamma^k)\Big((\chi(\gamma^i)+\chi(-1))\chi_\pi(\begin{pmatrix}\gamma^k & \\ & -\gamma^{k+i}\end{pmatrix}) +(1+\chi(-\gamma^i))\chi_\pi(\begin{pmatrix}-\gamma^k & \\ & \gamma^{k+i}\end{pmatrix})\Big)$
$C_3(i,k)$	$\chi(\gamma^k)\Big(1+\chi(\gamma^i)\Big)\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix})$
$C_4(i,k)$	$\chi(\gamma^k)\Big((\chi(\gamma^i)+\chi(\gamma^{-i}))\chi_\pi(\begin{pmatrix}\gamma^k & 1\\ & \gamma^k\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k-i}\end{pmatrix})\Big)$
$C_5(i,k)$	$\chi(\gamma^k)\Big((\chi(\gamma^i)+\chi(\gamma^{-i}))\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^k\end{pmatrix}) + (q+1)\chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k-i}\end{pmatrix})\Big)$
$C_6(i,j,k)$	$\chi(\gamma^k)\Big((1+\chi(\gamma^{i+j}))\chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k+j}\end{pmatrix}) + (\chi(\gamma^i)+\chi(\gamma^j))\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i+j}\end{pmatrix})\Big)$
$D_1(i,k)$	$\chi(\gamma^k)\Big(1+\chi(\gamma^i)\Big)\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix})$
$D_6(i,k)$	$(q+1)\chi(\gamma^k)\chi_\pi(\begin{pmatrix}\gamma^k\eta^i & \\ & \gamma^k\eta^{-i}\end{pmatrix})$
$D_8(i,k)$	$\chi(\gamma^k)\chi_\pi(\begin{pmatrix}\gamma^k\eta^i & \\ & \gamma^k\eta^{-i}\end{pmatrix})$

3.2 $\mathcal{G}$

The character table of $\mathcal{G}$ is the following.

Table 3.4: $\mathcal{G}$ character values

Conjugacy class	**$\mathcal{G}$ character value**
$A_1(q-1)$	$(q^4-1)(q^2-1)(q-1)$
$A_2(q-1)$	$-(q^2-1)(q-1)$
$A_{31}(q-1)$	$-(q^2-1)(q-1)$
$A_{32}(q-1)$	$-(q^2-1)(q-1)$
$A_5(q-1)$	$q-1$

We compute $(\chi_{\mathcal{G}}, \chi_{\mathcal{G}}) = q^2(q-1)$. So the number of irreducible generic representations of $\mathrm{GSp}(4, \mathbb{F}_q)$ is less than or equal to $q^2(q-1)$. It turns out, as we will soon see, that there are exactly that many such representations.

Chapter 4

Irreducible characters and cuspidality

We now determine all of the irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$ and determine if they are cuspidal or non–cuspidal. The complete list of irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$ is obtained from the results in [14] and [4]. After this list is given, we determine the irreducible non–cuspidal characters, i.e., the irreducible constituents of the characters induced from irreducible characters of the Borel, Siegel, and Klingen parabolic subgroups.

4.1 Irreducible characters

All of the irreducible characters of $\mathrm{Sp}(4, \mathbb{F}_q)$ were determined by Srinivasan in [14]. Her list of characters can be used to determine all of the irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$. A complete list of irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$ will help to determine the irreducible constituents of the parabolically induced irreducible cuspidal representations defined on the Borel, Siegel, and Klingen parabolic subgroups. It can then be said precisely which characters of $\mathrm{GSp}(4,\mathbb{F}_q)$ are irreducible cuspidal and which are irreducible non–cuspidal.

We have the following useful result from [4]. Let H be a subgroup of index 2 in a group G. Then H is a normal subgroup of G and G/H is a group of order 2. Let U and U' denote the trivial and non-trivial representations of G, respectively, obtained from the two representations of G/H. For any representation V of G, let $V' = V \otimes U'$. The character of V' is the same as the character of V on elements of H, but takes opposite values on elements not in H. In particular, $\mathrm{Res}_H^G V' = \mathrm{Res}_H^G V$. If W is any representation of H, there is a *conjugate* representation defined by conjugating by any element $t \in G$ such that $t \notin H$; if ψ is the character of W, the character of the conjugate is $h \mapsto \psi(tht^{-1})$. Since t is unique up to multiplication by an element of H, the conjugate representation is unique up to isomorphism.

Proposition 4.1.1. *Let V be an irreducible representation of G, and let $W = \mathrm{Res}_H^G V$ be the restriction of V to H. Then exactly one of the following holds:*

(1) *V is not isomorphic to V'; W is irreducible and isomorphic to its conjugate;* $\mathrm{Ind}_H^G W \cong V \oplus V'$.

(2) *$V \cong V'$; $W = W' \oplus W''$, where W' and W'' are irreducible and conjugate but not isomorphic;* $\mathrm{Ind}_H^G W' \cong \mathrm{Ind}_H^G W'' \cong V$.

Each irreducible representation of H arises uniquely in this way, noting that in case (1) *V and V' determine the same representation.*

The irreducible characters of $\mathrm{GSp}(4,\mathbb{F}_q)$ are determined as follows. Each irreducible representation of $\mathrm{Sp}(4,\mathbb{F}_q)$ is extended to a representation of $\mathrm{GSp}(4,\mathbb{F}_q)^+$, where $\mathrm{GSp}(4,\mathbb{F}_q)^+ := Z \cdot \mathrm{Sp}(4,\mathbb{F}_q)$. Note that $\mathrm{GSp}(4,\mathbb{F}_q)^+$ is an index two subgroup of $\mathrm{GSp}(4,\mathbb{F}_q)$ and that $Z \cap \mathrm{Sp}(4,\mathbb{F}_q) = \pm I$, where I is the identity of $\mathrm{GSp}(4,\mathbb{F}_q)$. For an irreducible representation π of $\mathrm{Sp}(4,\mathbb{F}_q)$, an irreducible representation π^+ of

$\mathrm{GSp}(4,\mathbb{F}_q)^+$ is constructed by defining $\pi^+(z \cdot g) := \alpha(z)\pi(g)$, where α is a character of Z, hence a character of $\mathbb{F}_q^\times$. By Schur's Lemma, elements of Z act as scalars on vectors in the space of π. To ensure that this new representation is well-defined, it is required that $\alpha(\pm 1)$ acts as $\pi(\pm I)$ on the space of π. The character of this new representation is $\alpha(z)\chi_\pi(g)$, where χ_π is the character of π. This representation is then induced to $\mathrm{GSp}(4,\mathbb{F}_q)$. The induced representation is either irreducible or has precisely two irreducible constituents.

The group $\mathrm{GSp}(4,\mathbb{F}_q)^+$ only has elements with square multipliers and so the induced character takes the value 0 on the non-square multiplier classes of $\mathrm{GSp}(4,\mathbb{F}_q)$. If the induced character decomposes into two constituents, then the sum of the values of the constituent characters on the non-square multiplier classes is 0. The values of the constituent characters on the square multiplier classes are half the values of the induced character on those classes.

Fix an isomorphism of $\mathbb{F}_{q^4}^\times$ into the multiplicative group $\mathbb{C}^\times$ and write the images of γ, θ, η, κ respectively as $\tilde{\gamma}$, $\tilde{\theta}$, $\tilde{\eta}$, $\tilde{\kappa}$. Define

$$t = \frac{1}{2}(q-1),$$

$$\alpha(j) = \tilde{\gamma}^j + \tilde{\gamma}^{-j},$$

$$\beta(j) = \tilde{\eta}^j + \tilde{\eta}^{-j}.$$

The table below lists all of the non-trivial irreducible characters, the value of $\alpha(-1)$, notation for the irreducible constituents of the induced character, and the dimension of each constituent. In the cases where the induced character decomposes, say into χ_a and χ_b, we have that $\chi_b = \xi\chi_a$, where $\xi : \mathbb{F}_q^\times \longrightarrow \mathbb{C}$ is the character defined by $\xi(\gamma) = -1$. Genericity of a character is indicated by a $\bullet$ in the "g" column.

Table 4.1: Irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$

Character	$\alpha(-1)$	**Constituents**	**Dimension**	**g**
$\mathrm{Ind}(\alpha\chi_1(n))$ $n \in R_1$	$(-1)^n$	$\mathrm{Ind}(\alpha\chi_1(n))_a$	$(q^2-1)^2$	•
		$\mathrm{Ind}(\alpha\chi_1(n))_b$	$(q^2-1)^2$	•
$\mathrm{Ind}(\alpha\chi_2(n))$ $n \in R_2$	$(-1)^n$	$\mathrm{Ind}(\alpha\chi_2(n))_a$	q^4-1	•
		$\mathrm{Ind}(\alpha\chi_2(n))_b$	q^4-1	•
$\mathrm{Ind}(\alpha\chi_3(n,m))$ $n, m \in T_1, n \neq m$	$(-1)^{n+m}$	$\mathrm{Ind}(\alpha\chi_3(n,m))_a$	$(q^2+1)(q+1)^2$	•
		$\mathrm{Ind}(\alpha\chi_3(n,m))_b$	$(q^2+1)(q+1)^2$	•
$\mathrm{Ind}(\alpha\chi_4(n,m))$ $n, m \in T_2, n < m$	$(-1)^{n+m}$	$\mathrm{Ind}(\alpha\chi_4(n,m))_a$	$(q^2+1)(q-1)^2$	•
		$\mathrm{Ind}(\alpha\chi_4(n,m))_b$	$(q^2+1)(q-1)^2$	•
$\mathrm{Ind}(\alpha\chi_5(n,m))$ $n \in T_2, m \in T_1$	$(-1)^{n+m}$	$\mathrm{Ind}(\alpha\chi_5(n,m))_a$	q^4-1	•
		$\mathrm{Ind}(\alpha\chi_5(n,m))_b$	q^4-1	•
$\mathrm{Ind}(\alpha\chi_6(n))$ $n \in T_2$	1	$\mathrm{Ind}(\alpha\chi_6(n))_a$	$(q^2+1)(q-1)$	
		$\mathrm{Ind}(\alpha\chi_6(n))_b$	$(q^2+1)(q-1)$	
$\mathrm{Ind}(\alpha\chi_7(n))$ $n \in T_2$	1	$\mathrm{Ind}(\alpha\chi_7(n))_a$	$q(q^2+1)(q-1)$	•
		$\mathrm{Ind}(\alpha\chi_7(n))_b$	$q(q^2+1)(q-1)$	•
$\mathrm{Ind}(\alpha\chi_8(n))$ $n \in T_1$	1	$\mathrm{Ind}(\alpha\chi_8(n))_a$	$(q^2+1)(q+1)$	
		$\mathrm{Ind}(\alpha\chi_8(n))_b$	$(q^2+1)(q+1)$	
$\mathrm{Ind}(\alpha\chi_9(n))$ $n \in T_1$	1	$\mathrm{Ind}(\alpha\chi_9(n))_a$	$q(q^2+1)(q+1)$	•
		$\mathrm{Ind}(\alpha\chi_9(n))_b$	$q(q^2+1)(q+1)$	•
$\mathrm{Ind}(\alpha\xi_1(n))$ $n \in T_2$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_1(n))_a$	$(q^2+1)(q-1)$	
		$\mathrm{Ind}(\alpha\xi_1(n))_b$	$(q^2+1)(q-1)$	

Table 4.1 – Continued

Character	$\alpha(-1)$	**Constituents**	**Dimension**	**g**
$\mathrm{Ind}(\alpha\xi_1'(n))$ $n \in T_2$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_1'(n))_a$	$q(q^2+1)(q-1)$	•
		$\mathrm{Ind}(\alpha\xi_1'(n))_b$	$q(q^2+1)(q-1)$	•
$\mathrm{Ind}(\alpha\xi_{21}(n))$ $n \in T_2$	$(-1)^{n+t}$	$\mathrm{Ind}(\alpha\xi_{21}(n))$	q^4-1	•
$\mathrm{Ind}(\alpha\xi_{21}'(n))$ $n \in T_2$	$(-1)^{n+t+1}$	$\mathrm{Ind}(\alpha\xi_{21}'(n))$	$(q^2+1)(q-1)^2$	•
$\mathrm{Ind}(\alpha\xi_3(n))$ $n \in T_1$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_3(n))_a$	$(q^2+1)(q+1)$	
		$\mathrm{Ind}(\alpha\xi_3(n))_b$	$(q^2+1)(q+1)$	
$\mathrm{Ind}(\alpha\xi_3'(n))$ $n \in T_1$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_3'(n))_a$	$q(q^2+1)(q+1)$	•
		$\mathrm{Ind}(\alpha\xi_3'(n))_b$	$q(q^2+1)(q+1)$	•
$\mathrm{Ind}(\alpha\xi_{41}(n))$ $n \in T_1$	$(-1)^{n+t}$	$\mathrm{Ind}(\alpha\xi_{41}(n))$	$(q^2+1)(q+1)^2$	•
$\mathrm{Ind}(\alpha\xi_{41}'(n))$ $n \in T_1$	$(-1)^{n+t+1}$	$\mathrm{Ind}(\alpha\xi_{41}'(n))$	q^4-1	•
$\mathrm{Ind}(\alpha\Phi_1)$	$(-1)^{t+1}$	$\mathrm{Ind}(\alpha\Phi_1)$	$(q^2+1)(q-1)$	
$\mathrm{Ind}(\alpha\Phi_3)$	$(-1)^{t+1}$	$\mathrm{Ind}(\alpha\Phi_3)$	$q(q^2+1)(q-1)$	•
$\mathrm{Ind}(\alpha\Phi_5)$	$(-1)^t$	$\mathrm{Ind}(\alpha\Phi_5)$	$(q^2+1)(q+1)$	
$\mathrm{Ind}(\alpha\Phi_7)$	$(-1)^t$	$\mathrm{Ind}(\alpha\Phi_7)$	$q(q^2+1)(q+1)$	•
$\mathrm{Ind}(\alpha\Phi_9)$	1	$\mathrm{Ind}(\alpha\Phi_9)_a$	$q(q^2+1)$	
		$\mathrm{Ind}(\alpha\Phi_9)_b$	$q(q^2+1)$	
$\mathrm{Ind}(\alpha\theta_1)$	1	$\mathrm{Ind}(\alpha\theta_1)$	$q^2(q^2+1)$	•

Table 4.1 – Continued

Character	$\alpha(-1)$	**Constituents**	**Dimension**	**g**
$\mathrm{Ind}(\alpha\theta_3)$	1	$\mathrm{Ind}(\alpha\theta_3)$	q^2+1	
$\mathrm{Ind}(\alpha\theta_5)$	-1	$\mathrm{Ind}(\alpha\theta_5)$	$q^2(q^2-1)$	•
$\mathrm{Ind}(\alpha\theta_7)$	-1	$\mathrm{Ind}(\alpha\theta_7)$	q^2-1	
$\mathrm{Ind}(\alpha\theta_9)$	1	$\mathrm{Ind}(\alpha\theta_9)_a$	$\frac{1}{2}q(q+1)^2$	
		$\mathrm{Ind}(\alpha\theta_9)_b$	$\frac{1}{2}q(q+1)^2$	
$\mathrm{Ind}(\alpha\theta_{10})$	1	$\mathrm{Ind}(\alpha\theta_{10})_a$	$\frac{1}{2}q(q-1)^2$	
		$\mathrm{Ind}(\alpha\theta_{10})_b$	$\frac{1}{2}q(q-1)^2$	
$\mathrm{Ind}(\alpha\theta_{11})$	1	$\mathrm{Ind}(\alpha\theta_{11})_a$	$\frac{1}{2}q(q^2+1)$	
		$\mathrm{Ind}(\alpha\theta_{11})_b$	$\frac{1}{2}q(q^2+1)$	
$\mathrm{Ind}(\alpha\theta_{12})$	1	$\mathrm{Ind}(\alpha\theta_{12})_a$	$\frac{1}{2}q(q^2+1)$	
		$\mathrm{Ind}(\alpha\theta_{12})_b$	$\frac{1}{2}q(q^2+1)$	
$\mathrm{Ind}(\alpha\theta_{13})$	1	$\mathrm{Ind}(\alpha\theta_{13})_a$	q^4	•
		$\mathrm{Ind}(\alpha\theta_{13})_b$	q^4	•

There are $(q^2+2q+4)(q-1)$ irreducible characters and $q^2(q-1)$ irreducible generic characters.

4.2 Irreducible non–cuspidal representations

Before the list of the irreducible non–cuspidal characters is given, we first give parts of the character tables of some irreducible non–cuspidal constituents $\sigma\mathrm{Ind}(\theta_1)$, $\sigma\mathrm{Ind}(\theta_3)$, $\mathrm{Ind}(\xi\omega_\pi\theta_5)$, $\mathrm{Ind}(\xi\omega_\pi\theta_7)$, $\sigma\mathrm{Ind}(\theta_9)$, $\sigma\mathrm{Ind}(\theta_{11})$, $\sigma\mathrm{Ind}(\theta_{12})$, $\mathrm{Ind}(\omega_\pi\Phi_1)$, $\mathrm{Ind}(\omega_\pi\Phi_3)$, $\mathrm{Ind}(\Phi_9)$, $\sigma\mathrm{St}_{\mathrm{GSp}(4)} := \sigma\mathrm{Ind}(\theta_{13})_a$, and $\sigma 1_{\mathrm{GSp}(4)}$.

Table 4.2: $\sigma\mathrm{Ind}(\chi_6(n))$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\chi_6(n))$ **character value**
$A_1(k)$	$2(q^2+1)(q-1)\sigma(\gamma^{2k})$
$A_2(k)$	$2(q-1)\sigma(\gamma^{2k})$
$A_{31}(k)$	$-2\sigma(\gamma^{2k})$
$A_{32}(k)$	$2(2q-1)\sigma(\gamma^{2k})$
$A_5(k)$	$-2\sigma(\gamma^{2k})$
$B_{11}(k)$	$2(-1)^{n+1}(q-1)^2\sigma(\gamma^{2k})$
$B_{12}(k)$	0
$B_{21}(k)$	$(1+(-1)^t)(q-1)\sigma(-\gamma^{2k})$
$B_{22}(k)$	$(1-(-1)^t)(q-1-\beta(\frac{q+1}{2}n))\sigma(-\gamma^{2k+1})$
$B_3(k)$	$2(-1)^n(q-1)\sigma(\gamma^{2k})$
$B_{41}(k)$	$2(-1)^{n+1}\sigma(\gamma^{2k})$
$B_{42}(k)$	$2(-1)^{n+1}\sigma(\gamma^{2k})$
$B_{43}(k)$	0
$B_{44}(k)$	0
$B_{51}(k)$	$-(1+(-1)^t)\sigma(-\gamma^{2k})$
$B_{52}(k)$	$-(1-(-1)^t)(\beta(\frac{q+1}{2}n)+1)\sigma(-\gamma^{2k+1})$
$C_1(i,k)$	$(1+(-1)^i)(q-1)\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	0
$C_{22}(i,k)$	$-2(1-(-1)^{t+i})\beta(\frac{q+1}{2}n)\sigma(-\gamma^{2k+i+1})$
$C_3(i,k)$	$-(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_4(i,k)$	0
$C_5(i,k)$	0

Table 4.2 – Continued

Conjugacy class	$\sigma\mathrm{Ind}(\chi_6(n))$ **character value**
$C_6(i,j,k)$	0
$D_2(i,k)$	$(1+(-1)^i)(q-1-\beta(in))\sigma(\gamma^{2k+i})$
$D_{31}(i,k)$	$-2(1+(-1)^{t+i})\beta(in)\sigma(-\gamma^{2k+i})$
$D_4(i,j,k)$	$-2(1+(-1)^{i+j})\beta(in)\sigma(\gamma^{2k+i+j})$
$D_5(i,k)$	$-(1+(-1)^i)(\beta(in)+1)\sigma(\gamma^{2k+i})$

Table 4.3: $\sigma\mathrm{Ind}(\chi_7(n))$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\chi_7(n))$ **character value**
$A_1(k)$	$2q(q-1)(q^2+1)\sigma(\gamma^{2k})$
$A_2(k)$	$2q(q-1)\sigma(\gamma^{2k})$
$A_{31}(k)$	$-2q\sigma(\gamma^{2k})$
$A_{32}(k)$	$-2q\sigma(\gamma^{2k})$
$A_5(k)$	0
$B_{11}(k)$	$2(-1)^n(q-1)^2\sigma(\gamma^{2k})$
$B_{12}(k)$	0
$B_{21}(k)$	$(1+(-1)^t)(q-1)\sigma(-\gamma^{2k})$
$B_{22}(k)$	$(1-(-1)^t)(1-q-q\beta(\frac{q+1}{2}n))\sigma(-\gamma^{2k+1})$
$B_3(k)$	$2(-1)^n(1-q)\sigma(\gamma^{2k})$
$B_{41}(k)$	$2(-1)^n\sigma(\gamma^{2k})$
$B_{42}(k)$	$2(-1)^n\sigma(\gamma^{2k})$

Table 4.3 – Continued

Conjugacy class	$\sigma\mathrm{Ind}(\chi_7(n))$ **character value**
$B_{43}(k)$	0
$B_{44}(k)$	0
$B_{51}(k)$	$-(1+(-1)^t)\sigma(-\gamma^{2k})$
$B_{52}(k)$	$(1-(-1)^t)\sigma(-\gamma^{2k+1})$
$C_1(i,k)$	$(1+(-1)^i)(q-1)\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	0
$C_{22}(i,k)$	$-2(1-(-1)^{t+i})\beta(\frac{q+1}{2}n)\sigma(-\gamma^{2k+i+1})$
$C_3(i,k)$	$-(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_4(i,k)$	0
$C_5(i,k)$	0
$C_6(i,j,k)$	0
$D_2(i,k)$	$(1+(-1)^i)(1-q-q\beta(in))\sigma(\gamma^{2k+i})$
$D_{31}(i,k)$	$-2(1+(-1)^{t+i})\beta(in)\sigma(-\gamma^{2k+i})$
$D_4(i,j,k)$	$-2(1+(-1)^{i+j})\beta(in)\sigma(\gamma^{2k+i+j})$
$D_5(i,k)$	$(1+(-1)^i)\sigma(\gamma^{2k+i})$

Table 4.4: $\sigma\mathrm{Ind}(\omega_\pi\Phi_1)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\omega_\pi\Phi_1)$ **character value**
$A_1(k)$	$(q^2+1)(q-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_2(k)$	$-(q^2-q+1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$

Table 4.4 – Continued

Conjugacy class	$\sigma\mathrm{Ind}(\omega_\pi\Phi_1)$ **character value**
$A_{31}(k)$	$(q-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_{32}(k)$	$(q-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_5(k)$	$-\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{11}(k)$	$(q-1)(1+(-1)^{t+1})\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{21}(k)$	0
$B_3(k)$	$(q+(-1)^t-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{41}(k)$	$((-1)^t-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{42}(k)$	$((-1)^t-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{51}(k)$	0
$C_1(i,k)$	0
$C_{21}(i,k)$	0
$C_3(i,k)$	0
$C_4(i,k)$	$-\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_5(i,k)$	$(q-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	0
$D_1(i,k)$	$(1+(-1)^i)\omega_\pi(\gamma^{k+i/2})\sigma(\gamma^{2k+i})$
$D_6(i,k)$	$(q+1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$D_8(i,k)$	$\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$

Table 4.5: $\sigma\mathrm{Ind}(\omega_\pi\Phi_3)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\omega_\pi\Phi_3)$ **character value**
$A_1(k)$	$q(q^2+1)(q-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_2(k)$	$-q\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_{31}(k)$	0
$A_{32}(k)$	$-2q\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_5(k)$	0
$B_{11}(k)$	$q(q-1)(1+(-1)^{t+1})\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{21}(k)$	0
$B_3(k)$	$q(-1)^t\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{41}(k)$	0
$B_{42}(k)$	0
$B_{51}(k)$	0
$C_1(i,k)$	0
$C_{21}(i,k)$	0
$C_3(i,k)$	0
$C_4(i,k)$	$-\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_5(i,k)$	$(q-1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	0
$D_1(i,k)$	$(1+(-1)^i)\omega_\pi(\gamma^{k+i/2})\sigma(\gamma^{2k+i})$
$D_6(i,k)$	$(q+1)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$D_8(i,k)$	$\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$

Table 4.6: $\sigma\mathrm{Ind}(\Phi_9)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\Phi_9)$ **character value**
$A_1(k)$	$2q(q^2+1)\sigma(\gamma^{2k})$
$A_2(k)$	$2q\sigma(\gamma^{2k})$
$A_{31}(k)$	$2q\sigma(\gamma^{2k})$
$A_{32}(k)$	$2q\sigma(\gamma^{2k})$
$A_5(k)$	0
$B_{11}(k)$	$2(-1)^t(q^2+1)\sigma(\gamma^{2k})$
$B_{21}(k)$	$(1+(-1)^t)(q+1)\sigma(-\gamma^{2k})$
$B_3(k)$	$2(-1)^t\sigma(\gamma^{2k})$
$B_{41}(k)$	$2(-1)^t\sigma(\gamma^{2k})$
$B_{42}(k)$	$2(-1)^t\sigma(\gamma^{2k})$
$B_{51}(k)$	$(1+(-1)^t)\sigma(-\gamma^{2k})$
$C_1(i,k)$	$(1+(-1)^i)(q+1)\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$2((-1)^i+(-1)^t)\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	$(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_4(i,k)$	$2(-1)^i\sigma(\gamma^{2k})$
$C_5(i,k)$	$2(-1)^i(q+1)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$2((-1)^i+(-1)^j)\sigma(\gamma^{2k+i+j})$

Table 4.7: $\sigma\mathrm{Ind}(\theta_1)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\theta_1)$ **character value**
$A_1(k)$	$q^2(q^2+1)\sigma(\gamma^{2k})$
$A_2(k)$	$q^2\sigma(\gamma^{2k})$
$A_{31}(k)$	0
$A_{32}(k)$	0
$A_5(k)$	0
$B_{11}(k)$	$2(-1)^t q\sigma(\gamma^{2k})$
$B_{21}(k)$	$(1+(-1)^t)q\sigma(-\gamma^{2k})$
$B_3(k)$	$(-1)^t q\sigma(\gamma^{2k})$
$B_{41}(k)$	0
$B_{42}(k)$	0
$B_{51}(k)$	0
$C_1(i,k)$	$(1+(-1)^i)q\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$((-1)^i+(-1)^t)\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	0
$C_4(i,k)$	$(-1)^i\sigma(\gamma^{2k})$
$C_5(i,k)$	$(-1)^i(q+1)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$((-1)^i+(-1)^j)\sigma(\gamma^{2k+i+j})$

Table 4.8: $\sigma\mathrm{Ind}(\theta_3)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\theta_3)$ **character value**
$A_1(k)$	$(q^2+1)\sigma(\gamma^{2k})$
$A_2(k)$	$\sigma(\gamma^{2k})$
$A_{31}(k)$	$(q+1)\sigma(\gamma^{2k})$
$A_{32}(k)$	$(1-q)\sigma(\gamma^{2k})$
$A_5(k)$	$\sigma(\gamma^{2k})$
$B_{11}(k)$	$2(-1)^t q\sigma(\gamma^{2k})$
$B_{21}(k)$	$(1+(-1)^t)\sigma(-\gamma^{2k})$
$B_3(k)$	$(-1)^t q\sigma(\gamma^{2k})$
$B_{41}(k)$	0
$B_{42}(k)$	0
$B_{51}(k)$	$(1+(-1)^t)\sigma(-\gamma^{2k})$
$C_1(i,k)$	$(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$((-1)^i+(-1)^t)\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	$(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_4(i,k)$	$(-1)^i\sigma(\gamma^{2k})$
$C_5(i,k)$	$(-1)^i(q+1)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$((-1)^i+(-1)^j)\sigma(\gamma^{2k+i+j})$

Table 4.9: $\sigma\mathrm{Ind}(\xi\omega_\pi\theta_5)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\xi\omega_\pi\theta_5)$ **character value**
$A_1(k)$	$q^2(q^2-1)\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_2(k)$	$-q^2\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_{31}(k)$	0
$A_{32}(k)$	0
$A_5(k)$	0
$B_{11}(k)$	0
$B_{21}(k)$	0
$B_3(k)$	$(-1)^t q\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{41}(k)$	0
$B_{42}(k)$	0
$B_{51}(k)$	0
$C_1(i,k)$	0
$C_{21}(i,k)$	0
$C_3(i,k)$	0
$C_4(i,k)$	$(-1)^{i+1}\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_5(i,k)$	$(q-1)(-1)^i\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	0
$D_1(i,k)$	$-(1+(-1)^i)\xi(\gamma^{k+i/2})\omega_\pi(\gamma^{k+i/2})\sigma(\gamma^{2k+i})$
$D_6(i,k)$	$-(q+1)(-1)^i\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$D_8(i,k)$	$(-1)^{i+1}\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$

Table 4.10: $\sigma\mathrm{Ind}(\xi\omega_\pi\theta_7)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\xi\omega_\pi\theta_7)$ **character value**
$A_1(k)$	$(q^2-1)\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_2(k)$	$-\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_{31}(k)$	$(q-1)\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_{32}(k)$	$-(q+1)\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$A_5(k)$	$-\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{11}(k)$	0
$B_{21}(k)$	0
$B_3(k)$	$(-1)^t q\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$B_{41}(k)$	0
$B_{42}(k)$	0
$B_{51}(k)$	0
$C_1(i,k)$	0
$C_{21}(i,k)$	0
$C_3(i,k)$	0
$C_4(i,k)$	$(-1)^{i+1}\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_5(i,k)$	$(q-1)(-1)^i\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	0
$D_1(i,k)$	$-(1+(-1)^i)\xi(\gamma^{k+i/2})\omega_\pi(\gamma^{k+i/2})\sigma(\gamma^{2k+i})$
$D_6(i,k)$	$(q+1)(-1)^{i+1}\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$
$D_8(i,k)$	$(-1)^{i+1}\xi(\gamma^k)\omega_\pi(\gamma^k)\sigma(\gamma^{2k})$

Table 4.11: $\sigma\mathrm{Ind}(\theta_9)$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\theta_9)$ **character value**
$A_1(k)$	$q(q+1)^2\sigma(\gamma^{2k})$
$A_2(k)$	$q(q+1)\sigma(\gamma^{2k})$
$A_{31}(k)$	$2q\sigma(\gamma^{2k})$
$A_{32}(k)$	0
$A_5(k)$	0
$B_{11}(k)$	$(q+1)^2\sigma(\gamma^{2k})$
$B_{21}(k)$	$(1+(-1)^t)(q+1)\sigma(-\gamma^{2k})$
$B_3(k)$	$(q+1)\sigma(\gamma^{2k})$
$B_{41}(k)$	$(q+1)\sigma(\gamma^{2k})$
$B_{42}(k)$	$(1-q)\sigma(\gamma^{2k})$
$B_{51}(k)$	$4(1+(-1)^t)\sigma(-\gamma^{2k})$
$C_1(i,k)$	$(1+(-1)^i)(q+1)\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$2(1+(-1)^{t+i})\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	$(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_4(i,k)$	$2\sigma(\gamma^{2k})$
$C_5(i,k)$	$2(q+1)\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$2(1+(-1)^{i+j})\sigma(\gamma^{2k+i+j})$

Table 4.12: $\sigma\mathrm{Ind}(\theta_{11})$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\theta_{11})$ **character value**
$A_1(k)$	$q(q^2+1)\sigma(\gamma^{2k})$
$A_2(k)$	$q(1-q)\sigma(\gamma^{2k})$
$A_{31}(k)$	$2q\sigma(\gamma^{2k})$
$A_{32}(k)$	0
$A_5(k)$	0
$B_{11}(k)$	$(q^2+2q-1)\sigma(\gamma^{2k})$
$B_{21}(k)$	$(1+(-1)^t)\sigma(-\gamma^{2k})$
$B_3(k)$	$(q-1)\sigma(\gamma^{2k})$
$B_{41}(k)$	$-(q+1)\sigma(\gamma^{2k})$
$B_{42}(k)$	$(q-1)\sigma(\gamma^{2k})$
$B_{51}(k)$	$(1+(-1)^t)\sigma(-\gamma^{2k})$
$C_1(i,k)$	$(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$(1+(-1)^{t+i})\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	$(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_4(i,k)$	0
$C_5(i,k)$	$2q\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$(1+(-1)^{i+j})\sigma(\gamma^{2k+i+j})$

Table 4.13: $\sigma\mathrm{Ind}(\theta_{12})$ character values

Conjugacy class	$\sigma\mathrm{Ind}(\theta_{12})$ **character value**
$A_1(k)$	$q(q^2+1)\sigma(\gamma^{2k})$
$A_2(k)$	$q(q+1)\sigma(\gamma^{2k})$
$A_{31}(k)$	0
$A_{32}(k)$	$2q\sigma(\gamma^{2k})$
$A_5(k)$	0
$B_{11}(k)$	$-(q^2-2q-1)\sigma(\gamma^{2k})$
$B_{21}(k)$	$q(1+(-1)^t)\sigma(-\gamma^{2k})$
$B_3(k)$	$(q+1)\sigma(\gamma^{2k})$
$B_{41}(k)$	$(1-q)\sigma(\gamma^{2k})$
$B_{42}(k)$	$(q+1)\sigma(\gamma^{2k})$
$B_{51}(k)$	0
$C_1(i,k)$	$q(1+(-1)^i)\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$(1+(-1)^{t+i})\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	0
$C_4(i,k)$	$2\sigma(\gamma^{2k})$
$C_5(i,k)$	$2\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$(1+(-1)^{i+j})\sigma(\gamma^{2k+i+j})$

Table 4.14: $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ character values

Conjugacy class	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ **character value**
$A_1(k)$	$q^4\sigma(\gamma^{2k})$
$A_2(k)$	0
$A_{31}(k)$	0
$A_{32}(k)$	0
$A_5(k)$	0
$B_{11}(k)$	$q^2\sigma(\gamma^{2k})$
$B_{21}(k)$	$\frac{1}{2}(1+(-1)^t)q\sigma(-\gamma^{2k})$
$B_3(k)$	0
$B_{41}(k)$	0
$B_{42}(k)$	0
$B_{51}(k)$	0
$C_1(i,k)$	$\frac{1}{2}(1+(-1)^i)q\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$\frac{1}{2}(1+(-1)^{t+i})\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	0
$C_4(i,k)$	0
$C_5(i,k)$	$q\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$\frac{1}{2}(1+(-1)^{i+j})\sigma(\gamma^{2k+i+j})$

Table 4.15: $\sigma 1_{\mathrm{GSp}(4)}$ character values

Conjugacy class	$\sigma 1_{\mathrm{GSp}(4)}$ **character value**
$A_1(k)$	$\sigma(\gamma^{2k})$
$A_2(k)$	$\sigma(\gamma^{2k})$
$A_{31}(k)$	$\sigma(\gamma^{2k})$
$A_{32}(k)$	$\sigma(\gamma^{2k})$
$A_5(k)$	$\sigma(\gamma^{2k})$
$B_{11}(k)$	$\sigma(\gamma^{2k})$
$B_{21}(k)$	$\sigma(-\gamma^{2k})$
$B_3(k)$	$\sigma(\gamma^{2k})$
$B_{41}(k)$	$\sigma(\gamma^{2k})$
$B_{42}(k)$	$\sigma(\gamma^{2k})$
$B_{51}(k)$	$\sigma(-\gamma^{2k})$
$C_1(i,k)$	$\sigma(\gamma^{2k+i})$
$C_{21}(i,k)$	$\sigma(-\gamma^{2k+i})$
$C_3(i,k)$	$\sigma(\gamma^{2k+i})$
$C_4(i,k)$	$\sigma(\gamma^{2k})$
$C_5(i,k)$	$\sigma(\gamma^{2k})$
$C_6(i,j,k)$	$\sigma(\gamma^{2k+i+j})$

The following table contains information on the irreducible non–cuspidal representations of the group $\mathrm{GSp}(4,\mathbb{F}_q)$. The format of the table is intentionally similar to a table in [11] with information on the irreducible, admissible, non–supercuspidal representations of $\mathrm{GSp}(4,\mathbb{F})$, where $\mathbb{F}$ is a non–archimedean local field.

Representations in the same group, denoted by a roman numeral, are constituents of the same induced representation. The group notation IV is not present. Since we are following the format of the related table in [11], we have tried to keep the group notations consistent. The representations associated to those given in the groups IV and VI in [11] are combined in the group IV*. The "g" column in the table indicates whether a representation is generic. We remark here that in the table the irreducible constituents are sometimes written as twists of an irreducible character rather than in the form that is in the table of irreducible characters above. For example, the Va constituent is written as $\sigma\mathrm{Ind}(\theta_1)$, where θ_1 is an irreducible character of $\mathrm{Sp}(4,\mathbb{F}_q)$ which is extended to $\mathrm{GSp}(4,\mathbb{F}_q)^+$ using the trivial character on the center, induced to $\mathrm{GSp}(4,\mathbb{F}_q)$, then twisted by the character σ. This constituent can also be written as $\mathrm{Ind}(\sigma^2\theta_1)$. The other such constituents can be written using the notation in the irreducible character table similarly.

Group I. These are the irreducible representations obtained by parabolic induction from the Borel subgroup. More precisely, they are irreducible representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$, with χ_1, χ_2, and σ characters of $\mathbb{F}_q^\times$. Representations of this form are irreducible if and only if $\chi_1 \neq 1_{\mathbb{F}_q^\times}, \chi_2 \neq 1_{\mathbb{F}_q^\times}$ and $\chi_1 \neq \chi_2^{\pm 1}$, where $1_{\mathbb{F}_q^\times}$ is the trivial character of $\mathbb{F}_q^\times$.

Group II. Let χ be a character of $\mathbb{F}_q^\times$ with $\chi^2 \neq 1_{\mathbb{F}_q^\times}$. Then the induced representation $\chi \times \chi \rtimes \sigma$ decomposes into two irreducible constituents

$$\mathrm{IIa}:\ \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma \quad \text{and} \quad \mathrm{IIb}:\ \chi 1_{\mathrm{GL}(2)} \rtimes \sigma,$$

where $\mathrm{St}_{\mathrm{GL}(2)}$ denotes the Steinberg representation of $\mathrm{GL}(2,\mathbb{F}_q)$ and $1_{\mathrm{GL}(2)}$ denotes the trivial representation of $\mathrm{GL}(2,\mathbb{F}_q)$. $\mathrm{St}_{\mathrm{GL}(2)}$ and $1_{\mathrm{GL}(2)}$ are obtained as the irreducible constituents of the induced representation $1_{\mathbb{F}_q^\times} \times 1_{\mathbb{F}_q^\times}$ of $\mathrm{GL}(2,\mathbb{F}_q)$.

Group III. Let χ be a character of $\mathbb{F}_q^\times$ such that $\chi \neq 1_{\mathbb{F}_q^\times}$. Then $\chi \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$ decomposes into two irreducible constituents

$$\text{IIIa}: \chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)} \quad \text{and} \quad \text{IIIb}: \chi \rtimes \sigma 1_{\mathrm{GSp}(2)},$$

where $\mathrm{St}_{\mathrm{GSp}(2)}$ denotes the Steinberg representation of $\mathrm{GSp}(2, \mathbb{F}_q)$ and $1_{\mathrm{GSp}(2)}$ denotes the trivial representation of $\mathrm{GSp}(2, \mathbb{F}_q)$. $\mathrm{St}_{\mathrm{GSp}(2)}$ and $1_{\mathrm{GSp}(2)}$ are obtained as the constituents of the induced representation $1_{\mathbb{F}_q^\times} \rtimes 1_{\mathbb{F}_q^\times}$ of $\mathrm{GSp}(2, \mathbb{F}_q)$.

Group V. Let ξ be a non-trivial quadratic character of $\mathbb{F}_q^\times$. Then $\xi \times \xi \rtimes \sigma$ decomposes into four irreducible constituents

$$\text{Va: } \sigma\mathrm{Ind}(\theta_1) \quad \text{Vb: } \sigma\mathrm{Ind}(\Phi_9)_a$$
$$\text{Vc: } \sigma\mathrm{Ind}(\Phi_9)_b \quad \text{Vd: } \sigma\mathrm{Ind}(\theta_3).$$

Group VI.* $1_{\mathbb{F}_q^\times} \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$ decomposes into six irreducible constituents

$$\text{VI*a: } \sigma\mathrm{St}_{\mathrm{GSp}(4)} \quad \text{VI*b: } \sigma\mathrm{Ind}(\theta_9)_a$$
$$\text{VI*c: } \sigma\mathrm{Ind}(\theta_9)_a \quad \text{VI*d: } \sigma\mathrm{Ind}(\theta_{11})_a$$
$$\text{VI*e: } \sigma\mathrm{Ind}(\theta_{12})_a \quad \text{VI*f: } \sigma 1_{\mathrm{GSp}(4)},$$

where $\mathrm{St}_{\mathrm{GSp}(4)} = \mathrm{Ind}(\theta_{13})$ is the Steinberg representation of $\mathrm{GSp}(4, \mathbb{F}_q)$ and $1_{\mathrm{GSp}(4)}$ is the trivial representation of $\mathrm{GSp}(4, \mathbb{F}_q)$.

Group VII. These are the irreducible representations of the form $\chi \rtimes \pi$, where π is an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ and χ is a character of $\mathbb{F}_q^\times$. Representations of this form are irreducible if and only if $\chi \neq 1_{\mathbb{F}_q^\times}$ and $\chi \neq \xi$, where ξ is a character of order 2 such that $\xi\pi \cong \pi$.

Group VIII. Let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with central character ω_π. Then $1_{\mathbb{F}_q^\times} \rtimes \pi$ decomposes into two irreducible constituents

$$\text{VIIIa: } \mathrm{Ind}(\omega_\pi\Phi_3) \quad \text{and} \quad \text{VIIIb: } \mathrm{Ind}(\omega_\pi\Phi_1).$$

Group IX. Let ξ be a non-trivial quadratic character of $\mathbb{F}_q^\times$ and let π be an irreducible cuspidal representation of $\mathrm{GL}(2,\mathbb{F}_q)$ with central character ω_π such that $\xi\pi \cong \pi$. Then $\xi \rtimes \pi$ decomposes into two irreducible constituents

$$\text{IXa: } \mathrm{Ind}(\xi\omega_\pi\theta_5) \quad \text{and} \quad \text{IXb: } \mathrm{Ind}(\xi\omega_\pi\theta_7).$$

Group X. These are the irreducible representations of the form $\pi \rtimes \sigma$, where π is an irreducible cuspidal representation of $\mathrm{GL}(2,\mathbb{F}_q)$ and σ is a character of $\mathbb{F}_q^\times$. Representations of this form are irreducible if and only if π does not have trivial central character ω_π.

Group XI. Let π be an irreducible cuspidal representation of $\mathrm{GL}(2,\mathbb{F}_q)$ with trivial central character ω_π and let σ be a character of $\mathbb{F}_q^\times$. Then $\pi \rtimes \sigma$ decomposes into two irreducible constituents

$$\text{XIa: } \sigma\mathrm{Ind}(\chi_7(n))_a \quad \text{and} \quad \text{XIb: } \sigma\mathrm{Ind}(\chi_6(n))_a.$$

Table 4.16: Irreducible non–cuspidal representations

		Constituent of	**Representation**	**Dimension**	**g**
I		$\chi_1 \times \chi_2 \rtimes \sigma$	(irreducible)	$(q^2+1)(q+1)^2$	•
II	a	$\chi \times \chi \rtimes \sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$q(q^2+1)(q+1)$	•
	b	$(\chi^2 \neq 1)$	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$(q^2+1)(q+1)$	
III	a	$\chi \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$q(q^2+1)(q+1)$	•
	b	$(\chi \neq 1)$	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$(q^2+1)(q+1)$	
V	a	$\xi \times \xi \rtimes \sigma$	$\sigma\mathrm{Ind}(\theta_1)$	$q^2(q^2+1)$	•
	b	$(\xi^2 = 1,\ \xi \neq 1)$	$\sigma\mathrm{Ind}(\Phi_9)_a$	$q(q^2+1)$	
	c		$\sigma\mathrm{Ind}(\Phi_9)_b$	$q(q^2+1)$	
	d		$\sigma\mathrm{Ind}(\theta_3)$	q^2+1	

Table 4.16 – Continued

		Constituent of	Representation	Dimension	g
VI*	a	$1_{\mathbb{F}_q^\times} \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	q^4	•
	b		$\sigma\mathrm{Ind}(\theta_9)_a$	$\frac{1}{2}q(q+1)^2$	
	c		$\sigma\mathrm{Ind}(\theta_9)_a$	$\frac{1}{2}q(q+1)^2$	
	d		$\sigma\mathrm{Ind}(\theta_{11})_a$	$\frac{1}{2}q(q^2+1)$	
	e		$\sigma\mathrm{Ind}(\theta_{12})_a$	$\frac{1}{2}q(q^2+1)$	
	f		$\sigma 1_{\mathrm{GSp}(4)}$	1	
VII		$\chi \rtimes \pi$ (irreducible)		q^4-1	•
VIII	a	$1_{\mathbb{F}_q^\times} \rtimes \pi$	$\mathrm{Ind}(\omega_\pi\Phi_3)$	$q(q^2+1)(q-1)$	•
	b		$\mathrm{Ind}(\omega_\pi\Phi_1)$	$(q^2+1)(q-1)$	
IX	a	$\xi \rtimes \pi$	$\mathrm{Ind}(\xi\omega_\pi\theta_5)$	$q^2(q^2-1)$	•
	b	$(\xi \neq 1,\ \xi\pi = \pi)$	$\mathrm{Ind}(\xi\omega_\pi\theta_7)$	q^2-1	
X		$\pi \rtimes \sigma$ (irreducible)		q^4-1	•
XI	a	$\pi \rtimes \sigma$	$\sigma\mathrm{Ind}(\chi_7(n))_a$	$q(q^2+1)(q-1)$	•
	b	$(\omega_\pi = 1)$	$\sigma\mathrm{Ind}(\chi_6(n))_a$	$(q^2+1)(q-1)$	

The induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq 1$, $\chi_2 \neq 1$, and $\chi_1 \neq \chi_2^{\pm 1}$.

Several formulas are used in the proof of this table. In particular, they are used to establish reducibility criteria and to verify that a particular representation is a constituent of a non–cuspidal group of representations. Define

$$z_1 = \begin{cases} 0 & \text{if } \chi_1 \neq 1 \\ q-1 & \text{if } \chi_1 = 1 \end{cases} \qquad z_2 = \begin{cases} 0 & \text{if } \chi_2 \neq 1 \\ q-1 & \text{if } \chi_2 = 1 \end{cases}$$

$$z_3 = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_1^{-1} \\ q-1 & \text{if } \chi_1 = \chi_1^{-1} \end{cases} \qquad z_4 = \begin{cases} 0 & \text{if } \chi_2 \neq \chi_2^{-1} \\ q-1 & \text{if } \chi_2 = \chi_2^{-1} \end{cases}$$

$$z_5 = \begin{cases} 0 & \text{if } \chi_2 \neq \chi_1 \\ q-1 & \text{if } \chi_2 = \chi_1 \end{cases} \qquad z_6 = \begin{cases} 0 & \text{if } \chi_2 \neq \chi_1^{-1} \\ q-1 & \text{if } \chi_2 \neq \chi_1^{-1} \end{cases}$$

$$z_7 = \begin{cases} 0 & \text{if } \phi|_{\langle \eta \rangle} \neq 1 \\ q+1 & \text{if } \phi|_{\langle \eta \rangle} = 1. \end{cases}$$

Lemma 4.2.1. *Let χ_1, $\chi_2 : \mathbb{F}_q^\times \longrightarrow \mathbb{C}$ be characters. Then*

$$\sum_{i \in T_1} \chi_1(\gamma^i) + \chi_1(\gamma^{-i}) = z_1 - 1 - \chi_1(-1)$$

$$\sum_{i \in T_1} \chi_1(\gamma^i)\chi_2(\gamma^i) + \chi_1(\gamma^{-i})\chi_2(\gamma^{-i}) = z_6 - 1 - \chi_1(-1)\chi_2(-1)$$

$$\sum_{i \in T_1} \chi_1(\gamma^i)\chi_2(\gamma^{-i}) + \chi_1(\gamma^{-i})\chi_2(\gamma^i) = z_5 - 1 - \chi_1(-1)\chi_2(-1)$$

$$\sum_{i \in T_1} \chi_1(\gamma^i)^2 + \chi_1(\gamma^{-i})^2 = z_3 - 2$$

$$\sum_{i,j\in T_1, i<j} \chi_1(\gamma^{i+j}) + \chi_1(\gamma^{i-j}) + \chi_1(\gamma^{-i+j}) + \chi_1(\gamma^{-i-j})$$

$$= \frac{1}{2}\big((z_1 - \chi_1(-1) - 1)^2 - z_3 - q + 5\big)$$

$$\sum_{i,j\in T_1, i<j} (\chi_1(\gamma^i) + \chi_1(\gamma^{-i}))(\chi_2(\gamma^i) + \chi_2(\gamma^{-i}))$$

$$= \frac{q-5}{4}\,(z_6 + z_5 - 2\chi_1(-1)\chi_2(-1) - 2)$$

$$\sum_{i,j\in T_1, i<j} \chi_1(\gamma^i) + \chi_1(\gamma^{-i}) + \chi_1(\gamma^j) + \chi_1(\gamma^{-j}) = \frac{q-5}{2}\,(z_1 - 1 - \chi_1(-1))$$

$$\sum_{i,j\in T_1, i<j} \chi_1(\gamma^i)^2 + \chi_1(\gamma^{-i})^2 + \chi_1(\gamma^j)^2 + \chi_1(\gamma^{-j})^2 = \frac{q-5}{2}\,(z_3 - 2)$$

Proof: Straightforward. □

Now we prove the assertions in the above table of irreducible non–cuspidal representations.

Proof: The irreducible non–cuspidal representations are supported in the Borel, the Siegel parabolic, or the Klingen parabolic. We first consider those supported in the Borel.

Borel: Let χ_1, χ_2, and σ be characters of $\mathbb{F}_q^\times$. As in 3.1.1, these characters are used to define a representation of the Borel subgroup and induced to $\mathrm{GSp}(4, \mathbb{F}_q)$ to obtain the representation $\chi_1 \times \chi_2 \rtimes \sigma$. From its character table, we have

$$\chi_1 \times \chi_2 \rtimes \sigma \cong \chi_2 \times \chi_1 \rtimes \sigma.$$

We also have

$$(\chi_1 \times \chi_2 \rtimes \sigma,\, \chi_{\mathcal{G}}) = 1$$

for all such representations $\chi_1 \times \chi_2 \rtimes \sigma$, indicating that exactly one irreducible constituent of $\chi_1 \times \chi_2 \rtimes \sigma$ is generic. Also,

$$(\chi_1 \times \chi_2 \rtimes \sigma,\, \chi_1 \times \chi_2 \rtimes \sigma) = \frac{q^2 + (z_5 + z_6 - 2)q + 1 + z_1^2 + 2z_1z_2 + z^2 - z_5 - z_6 + z_5z_6}{(q-1)^2} \tag{4.1}$$

which has precisely four possible values: 1,2,4,8. Equation 4.1 is equal to 1, or, equivalently, $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible, if and only if $\chi_1 \neq 1$, $\chi_2 \neq 1$, and $\chi_2 \neq \chi_1^{\pm 1}$.

Equation 4.1 is equal to 2 if one of the following holds:

1. $\chi_1 \neq \chi_1^{-1}$, $\chi_2 = \chi_1^{-1}$
2. $\chi_1 \neq \chi_1^{-1}$, $\chi_2 = \chi_1$
3. $\chi_1 \neq \chi_1^{-1}$, $\chi_2 = 1$
4. $\chi_1 = \chi_1^{-1}$, $\chi_1 \neq 1$, $\chi_2 = 1$
5. $\chi_1 = 1$, $\chi_2 \neq \chi_2^{-1}$
6. $\chi_1 = 1$, $\chi_2 = \chi_2^{-1}$, $\chi_2 \neq 1$

Equation 4.1 is equal to 4 if and only if $\chi_1 = \chi_2$, $\chi_2 \neq 1$, $\chi_1 = \chi_1^{-1}$.

Equation 4.1 is equal to 8 if and only if $\chi_1 = 1$ and $\chi_2 = 1$.

Using the character inner product, the constituents in groups II–IV* can be verified.

Siegel: Let σ be a character of $\mathbb{F}_q^\times$ and let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with central character ω_π. As in 3.1.2, define a representation of the Siegel parabolic subgroup and induce to $\mathrm{GSp}(4, \mathbb{F}_q)$ to obtain the representation $\pi \rtimes \sigma$. We have

$$(\pi \rtimes \sigma,\, \chi_{\mathcal{G}}) = 1$$

for all such representations $\pi \rtimes \sigma$, indicating that exactly one irreducible constituent of $\pi \rtimes \sigma$ is generic.

The sum of the dimensions of the irreducible constituents of $\pi \rtimes \sigma$ is $q^4 - 1$. If $\pi \rtimes \sigma$ is reducible, then by dimension considerations, the possible irreducible generic constituents are $\mathrm{Ind}(\alpha\chi_1(n))_a$, $\mathrm{Ind}(\alpha\chi_1(n))_b$, $\mathrm{Ind}(\alpha\chi_4(n,m))_a$, $\mathrm{Ind}(\alpha\chi_4(n,m))_b$, $\mathrm{Ind}(\alpha\chi_7(n))_a$, $\mathrm{Ind}(\alpha\chi_7(n))_b$, $\mathrm{Ind}(\alpha\xi_1'(n))_a$, $\mathrm{Ind}(\alpha\xi_1'(n))_b$, $\mathrm{Ind}(\alpha\xi_{21}'(n))$, $\mathrm{Ind}(\alpha\Phi_3)$, and $\mathrm{Ind}(\alpha\theta_5)$.

By adding character values, if $\pi \rtimes \sigma$ is reducible then it has precisely two irreducible constituents, ω_π is trivial, and

$$\pi \rtimes \sigma = \sigma\mathrm{Ind}(\chi_7(n))_a + \sigma\mathrm{Ind}(\chi_6(n))_a$$

for some $n \in T_2$.

Klingen: Let χ be a character of $\mathbb{F}_q^\times$ and let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with central character ω_π. As in 3.1.3, define a representation of the Klingen parabolic subgroup and induce to $\mathrm{GSp}(4, \mathbb{F}_q)$ to obtain the representation $\chi \rtimes \pi$. We have

$$(\chi \rtimes \pi,\ \chi_{\mathcal{G}}) = 1$$

for all such representations $\chi \rtimes \pi$, indicating that exactly one irreducible constituent of $\chi \rtimes \pi$ is generic.

The sum of the dimensions of the irreducible constituents of $\chi \rtimes \pi$ is $q^4 - 1$. If $\chi \rtimes \pi$ is reducible, the possible irreducible generic constituents are $\mathrm{Ind}(\alpha\chi_1(n))_a$, $\mathrm{Ind}(\alpha\chi_1(n))_b$, $\mathrm{Ind}(\alpha\chi_4(n,m))_a$, $\mathrm{Ind}(\alpha\chi_4(n,m))_b$, $\mathrm{Ind}(\alpha\chi_7(n))_a$, $\mathrm{Ind}(\alpha\chi_7(n))_b$, $\mathrm{Ind}(\alpha\xi_1'(n))_a$, $\mathrm{Ind}(\alpha\xi_1'(n))_b$, $\mathrm{Ind}(\alpha\xi_{21}'(n))$, $\mathrm{Ind}(\alpha\Phi_3)$, and $\mathrm{Ind}(\alpha\theta_5)$.

By adding character values, if $\chi \rtimes \pi$ is reducible then it has precisely two irreducible constituents and either

(1) χ is trivial and

$$1_{\mathbb{F}_q^\times} \rtimes \pi = \mathrm{Ind}(\omega_\pi \Phi_3) + \mathrm{Ind}(\omega_\pi \Phi_1)$$

or (2) $\chi = \xi$ with $\xi \neq 1$ such that $\xi\pi = \pi$ and

$$\xi \rtimes \pi = \mathrm{Ind}(\xi\omega_\pi \theta_5) + \mathrm{Ind}(\xi\omega_\pi \theta_7).$$

$\square$

4.2.1 Decompositions for types V and VI*

We will now give more detailed descriptions of the decompositions of the non–cuspidal representations supported in the Borel subgroup for types V and VI*. These decompositions can be verified with the character tables provided above using either the inner product or by adding up character values on each conjugacy class.

Group V: Constituents of $\xi \times \xi \rtimes \sigma$, where ξ is a non-trivial quadratic character.

$$\begin{aligned} \xi \times \xi \rtimes \sigma &= \xi\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma + \xi\, 1_{\mathrm{GL}(2)} \rtimes \sigma \\ &= \xi\, \mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi\sigma + \xi\, 1_{\mathrm{GL}(2)} \rtimes \xi\sigma. \end{aligned}$$

Each of the four representations on the right side is reducible and has two constituents as shown in the following table.

Table 4.17: Group V constituents

	$\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi\sigma$	$\xi\,1_{\mathrm{GL}(2)} \rtimes \xi\sigma$
$\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\sigma\mathrm{Ind}(\theta_1)$	$\sigma\mathrm{Ind}(\Phi_9)_a$
$\xi\,1_{\mathrm{GL}(2)} \rtimes \sigma$	$\sigma\mathrm{Ind}(\Phi_9)_b$	$\sigma\mathrm{Ind}(\theta_3)$

Group VI*: Constituents of $1_{\mathbb{F}^\times} \times 1_{\mathbb{F}^\times} \rtimes \sigma$.

$$\begin{aligned} 1_{\mathbb{F}^\times} \times 1_{\mathbb{F}^\times} \rtimes \sigma &= \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma + 1_{\mathrm{GL}(2)} \rtimes \sigma \\ &= 1_{\mathbb{F}_q^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)} + 1_{\mathbb{F}_q^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}. \end{aligned}$$

Each of the four representations on the right is reducible and has three irreducible constituents as shown in the following table. The common factor $\sigma\mathrm{Ind}(\theta_9)_a$ occurs as a constituent of each of the four representations $\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$, $1_{\mathrm{GL}(2)} \rtimes \sigma$, $1_{\mathbb{F}_q^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$, and $1_{\mathbb{F}_q^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}$.

Table 4.18: Group VI* constituents

	$\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	(common factor)	$1_{\mathrm{GL}(2)} \rtimes \sigma$
$1_{\mathbb{F}_q^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\sigma\mathrm{Ind}(\theta_9)_a$	$\sigma\mathrm{Ind}(\theta_{11})_a$
(common factor)	$\sigma\mathrm{Ind}(\theta_9)_a$	–	$\sigma\mathrm{Ind}(\theta_9)_a$
$1_{\mathbb{F}_q^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$\sigma\mathrm{Ind}(\theta_{12})_a$	$\sigma\mathrm{Ind}(\theta_9)_a$	$\sigma 1_{\mathrm{GSp}(4)}$

4.3 Irreducible non–supercuspidal representations

For comparison purposes, we now summarize some results of Sally and Tadić in [12] on the irreducible admissible non–supercuspidal representations of $\mathrm{GSp}(4, F)$, where F is a non–archimedean local field of characteristic 0.

The following table from [11] lists all such representations. As in our previous non–cuspidal table, representations in the same group I – XI are constituents of the same induced representation. The "g" column indicates the generic representations.

Representations in groups I – VI are constituents of representations induced from a character of the Borel subgroup $B(F)$. Representations in groups VII – IX are supported in Q, and representations in groups X and XI are supported in P. Let π be a supercuspidal representation of $\mathrm{GL}(2, F)$ with central character ω_π.

Table 4.19: Irreducible non–supercuspidal representations

		Constituent of	**Representation**	**g**
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)		•
II	a	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	•
	b	$(\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2})$	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	
III	a	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	•
	b	$(\chi \notin \{1, \nu^{\pm 2}\})$	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	
IV	a	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	•
	b		$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	
	c		$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	
	d		$\sigma 1_{\mathrm{GSp}(4)}$	

Table 4.19 – Continued

		Constituent of	**Representation**	**g**
V	a	$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\bullet$
	b	$(\xi^2 = 1,\ \xi \neq 1)$	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	
	c		$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	
	d		$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	
VI	a	$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$	$\tau(S, \nu^{-1/2}\sigma)$	$\bullet$
	b		$\tau(T, \nu^{-1/2}\sigma)$	
	c		$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	
	d		$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	
VII		$\chi \rtimes \pi$ (irreducible)		$\bullet$
VIII	a	$1_{F^\times} \rtimes \pi$	$\tau(S, \pi)$	$\bullet$
	b		$\tau(T, \pi)$	
IX	a	$\nu\xi \rtimes \nu^{-1/2}\pi$	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$\bullet$
	b	$(\xi \neq 1,\ \xi\pi = \pi)$	$L(\nu\xi, \nu^{-1/2}\pi)$	
X		$\pi \rtimes \sigma$ (irreducible)		$\bullet$
XI	a	$\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\bullet$
	b	$(\omega_\pi = 1)$	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	

The induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq \nu^{\pm 1}$, $\chi_2 \neq \nu^{\pm 1}$, and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$.

4.3.1 Decompositions for types IV, V, and VI

Group IV: Constituents of $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$.

$$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma = \underbrace{\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma}_{\text{sub}} + \underbrace{\nu^{3/2}1_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma}_{\text{quot}}$$
$$= \underbrace{\nu^2 \rtimes \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}}_{\text{sub}} + \underbrace{\nu^2 \rtimes \nu^{-1}\sigma 1_{\mathrm{GSp}(2)}}_{\text{quot}}.$$

Each of the four representations on the right is reducible and has two irreducible constituents as shown in the following table. The quotients are on the bottom, respectively on the right.

Table 4.20: Group IV constituents

	$\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$	$\nu^{3/2}1_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$
$\nu^2 \rtimes \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$
$\nu^2 \rtimes \nu^{-1}\sigma 1_{\mathrm{GSp}(2)}$	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\sigma 1_{\mathrm{GSp}(4)}$

Group V: Constituents of $\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$, where ξ is a non-trivial quadratic character.

$$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma = \underbrace{\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\xi\,1_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{quot}}$$
$$= \underbrace{\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\xi\,1_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}\sigma}_{\text{quot}}.$$

Each of the representations on the right side has two constituents as indicated in the following table. The quotients appear on the bottom, respectively on the right.

Table 4.21: Group V constituents

	$\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}\sigma$	$\nu^{1/2}\xi\,1_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}\sigma$
$\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$
$\nu^{1/2}\xi\,1_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$L(\nu\xi,\xi \rtimes \nu^{-1/2}\sigma)$

Group VI: Constituents of $\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$.

$$\begin{aligned}\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma &= \underbrace{\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\,1_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{quot}} \\ &= \underbrace{1_{F^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}}_{\text{sub}} + \underbrace{1_{F^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}}_{\text{quot}}.\end{aligned}$$

Each of the representations on the right side is again reducible, having two constituents as indicated in the following table. The quotients appearing on the bottom, respectively on the right.

Table 4.22: Group VI constituents

	$\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$\nu^{1/2}\,1_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$
$1_{F^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\tau(S,\nu^{-1/2}\sigma)$	$\tau(T,\nu^{-1/2}\sigma)$
$1_{F^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(\nu,1_{F^\times} \rtimes \nu^{-1/2}\sigma)$

4.4 Subspaces of admissible non–supercuspidal representations

Consider the group $\mathrm{GSp}(4,F)$, where F is a non–archimedean local field of characteristic zero. Denote its ring of integers by $\mathfrak{o}$ and let $\mathfrak{p}$ be its maximal ideal. Consider only those fields F such that $\mathfrak{o}/\mathfrak{p}$ is isomorphic to the finite field $\mathbb{F}_q$ so that $\mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p}) \cong \mathrm{GSp}(4,\mathbb{F}_q)$. Fix a generator ϖ of $\mathfrak{p}$. If x is in $F^\times$, then define $\nu(x)$ to be the unique integer such that $x = u\varpi^{\nu(x)}$ for some unit u in $\mathfrak{o}^\times$. Write $\nu(x)$ or $|x|$ for the normalized absolute value of x; thus $\nu(\varpi) = q^{-1}$.

Recall that a representation (π, V) of a group G is called *smooth* if every vector in V is fixed by an open-compact subgroup K of G and (π, V) is called *admissible* if the spaces V^K of fixed vectors under the action of an open-compact subgroup K are finite dimensional for any open compact subgroup K.

The *congruence subgroup of level* $\mathfrak{p}^n$ of $\mathrm{GSp}(4,F)$, denoted by $\Gamma(\mathfrak{p}^n)$, is defined by

$$\Gamma(\mathfrak{p}^n) = \{g \in \mathrm{GSp}(4,F) : g \equiv I \pmod{\mathfrak{p}^n}\}$$

where I is the 4×4 identity matrix.

We have the following short exact sequence

$$1 \longrightarrow \Gamma(\mathfrak{p}) \longrightarrow \mathrm{GSp}(4,\mathfrak{o}) \longrightarrow \mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p}) \longrightarrow 1$$

for the maximal compact subgroup $K = \mathrm{GSp}(4,\mathfrak{o})$ of $\mathrm{GSp}(4,F)$.

Let (π, V) be an admissible representation of $\mathrm{GSp}(4,F)$. K acts on the space $V^{\Gamma(\mathfrak{p})}$. By definition $\Gamma(\mathfrak{p})$ acts trivially in this space so there is an action of the group $\mathrm{GSp}(4,\mathbb{F}_q) \cong \mathrm{GSp}(4,\mathfrak{o}/\mathfrak{p}) \cong K/\Gamma(\mathfrak{p})$.

For an admissible non–supercuspidal representation of $\mathrm{GSp}(4,F)$, the dimensions of the spaces $V^{\Gamma(\mathfrak{p})}$ can be determined by looking at their finite group analogues. For example, consider the irreducible non–supercuspidal representation $\chi_1 \times \chi_2 \rtimes \sigma$ given in [12]. The standard model for this representation is the space of smooth functions

$$f : \mathrm{GSp}(4,F) \longrightarrow \mathbb{C}$$

satisfying

$$f(\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} g) = |a^2 b||c|^{-3/2} \chi_1(a)\chi_2(b)\sigma(c) f(g),$$

with group action by right translation and each such f fixed by some open-compact subgroup of $\mathrm{GSp}(4,F)$. χ_1, χ_2, and σ are characters of the multiplicative group of the field F, i.e., $\chi_1, \chi_2, \sigma : F^\times \longrightarrow \mathbb{C}^\times$. Let us assume that these characters are trivial on $1+\mathfrak{p}$. We restrict these characters to $\mathfrak{o}^\times$ to get characters on the multiplicative group of the finite field $\mathbb{F}_q$, i.e.,

$$\tilde{\chi_i} := \chi_i|_{\mathfrak{o}^\times}, \tilde{\sigma} := \sigma|_{\mathfrak{o}^\times} : \mathbb{F}_q^\times \cong (\mathfrak{o}/\mathfrak{p})^\times = \mathfrak{o}^\times/(1+\mathfrak{p}) \longrightarrow \mathbb{C}^\times.$$

The subspace of this representation $V^{\Gamma(\mathfrak{p})}$ of vectors fixed under the action of the congruence subgroup $\Gamma(\mathfrak{p})$

$$V^{\Gamma(\mathfrak{p})} = \{f : f(gk) = f(g) \,\mathrm{for\,all}\, k \in \Gamma(\mathfrak{p})\}$$

is isomorphic to the space $\tilde{V}^{\Gamma(\mathfrak{p})}$ of functions $f : K \longrightarrow \mathbb{C}$ satisfying the above property. These functions are then functions $f : K/\Gamma(\mathfrak{p}) \longrightarrow \mathbb{C}$, hence functions

$f : \mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p}) \longrightarrow \mathbb{C}$. So f can be considered to be a function on the finite group $\mathrm{GSp}(4, \mathbb{F}_q)$ satisfying

$$f(\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} g) = \tilde{\chi_1}(a)\tilde{\chi_2}(b)\tilde{\sigma}(c)f(g).$$

So

$$(\chi_1 \times \chi_2 \rtimes \sigma)^{\Gamma(\mathfrak{p})} \cong \tilde{\chi_1} \times \tilde{\chi_2} \rtimes \tilde{\sigma}.$$

Thus the dimension of $(\chi_1 \times \chi_2 \rtimes \sigma)^{\Gamma(\mathfrak{p})}$ is $(q^2+1)(q+1)^2$.

The non–supercuspidal representations obtained from parabolic induction on the Siegel and Klingen parabolic groups also descend to non–cuspidal representations in the finite group case under appropriate assumptions on the characters σ and χ and the GL(2) representation π, i.e., σ and χ are trivial on $1 + \mathfrak{p}$, and π has non-zero $\Gamma(\mathfrak{p})$–fixed vectors for the principal congruence subgroup $\Gamma(\mathfrak{p})$ of $\mathrm{GL}(2, F)$.

The following table gives the dimensions of $\Gamma(\mathfrak{p})$–fixed vectors in the non-supercuspidal characters supported in the Borel, the Siegel parabolic, or the Klingen parabolic subgroup. The "$\mathrm{GSp}(4, \mathbb{F}_q)$" column indicates the finite representations producing the dimensions given.

Table 4.23: Dimensions of $\Gamma(\mathfrak{p})$–fixed vectors

		Representation	**Dimension**	$\mathrm{GSp}(4, \mathbb{F}_q)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$(q^2+1)(q+1)^2$	$\chi_1 \times \chi_2 \rtimes \sigma$
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$q(q^2+1)(q+1)$	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$(q^2+1)(q+1)$	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$

Table 4.23 – Continued

		Representation	**Dimension**	$\mathrm{GSp}(4,\mathbb{F}_q)$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$q(q^2+1)(q+1)$	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$(q^2+1)(q+1)$	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	q^4	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$q(q^2+q+1)$	$\sigma\mathrm{Ind}(\theta_9)_a + \sigma\mathrm{Ind}(\theta_{11})_a$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$q(q^2+q+1)$	$\sigma\mathrm{Ind}(\theta_9)_a + \sigma\mathrm{Ind}(\theta_{12})_a$
	d	$\sigma 1_{\mathrm{GSp}(4)}$	1	$\sigma 1_{\mathrm{GSp}(4)}$
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$q^2(q^2+1)$	$\sigma\mathrm{Ind}(\theta_1)$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$q(q^2+1)$	$\sigma\mathrm{Ind}(\Phi_9)_a$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$q(q^2+1)$	$\sigma\mathrm{Ind}(\Phi_9)_b$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	q^2+1	$\sigma\mathrm{Ind}(\theta_3)$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$q^4 + \frac{1}{2}q(q+1)^2$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)} + \sigma\mathrm{Ind}(\theta_9)_a$
	b	$\tau(T, \nu^{-1/2}\sigma)$	$\frac{1}{2}q(q^2+1)$	$\sigma\mathrm{Ind}(\theta_{11})_a$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\frac{1}{2}q(q^2+1)$	$\sigma\mathrm{Ind}(\theta_{12})_a$
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$1 + \frac{1}{2}q(q+1)^2$	$\sigma 1_{\mathrm{GSp}(4)} + \sigma\mathrm{Ind}(\theta_9)_a$
VII		$\chi \rtimes \pi$ (irreducible)	q^4-1	$\chi \rtimes \pi$
VIII	a	$\tau(S, \pi)$	$q(q^2+1)(q-1)$	$\mathrm{Ind}(\omega_\pi\Phi_3)$
	b	$\tau(T, \pi)$	$(q^2+1)(q-1)$	$\mathrm{Ind}(\omega_\pi\Phi_1)$
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$q^2(q^2-1)$	$\mathrm{Ind}(\xi\omega_\pi\theta_5)$
	b	$L(\nu\xi, \nu^{-1/2}\pi)$	q^2-1	$\mathrm{Ind}(\xi\omega_\pi\theta_7)$
X		$\pi \rtimes \sigma$ (irreducible)	q^4-1	$\pi \rtimes \sigma$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$q(q^2+1)(q-1)$	$\sigma\mathrm{Ind}(\chi_7(n))_a$
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$(q^2+1)(q-1)$	$\sigma\mathrm{Ind}(\chi_6(n))_a$

The dimensions in the table can be verified by comparing the decompositions of each type to the finite group decompositions of the corresponding type. Note that we may assume that $\sigma = 1$ since the space of fixed vectors doesn't change under twisting with characters trivial on $1 + \mathfrak{p}$. Indeed, let π be a representation of $\mathrm{GSp}(4, F)$, σ a character of $F^\times$ which is trivial on $1 + \mathfrak{p}$, and v a $\Gamma(\mathfrak{p})$–fixed vector. Then, for all $g \in \Gamma(\mathfrak{p})$, we have

$$(\sigma\pi)(g)v = \sigma(\lambda(g))\pi(g)v = \sigma(\lambda(g))v = v.$$

The type VI representations require some additional information to compute dimensions using Table 4.18. The issue is where to place the common factor $\sigma\mathrm{Ind}(\theta_9)_a$. It is important to note that VId has a three–dimensional subspace of Iwahori subgroup–fixed vectors so the dimension of its space of $\Gamma(\mathfrak{p})$–fixed vectors is at least three–dimensional. Comparing with Table 4.18 completes the argument for the dimensions.

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